AN

ELEMENTARY TREATISE

ON

ARITHMETIC,

TAKEN

PRINCIPALLY FROM THE ARITHMETIC

OF

S. F. LACROIX,

AND

TRANSLATED INTO ENGLISH WITH SUCH ALTERATIONS AND ADDITIONS AS WERE FOUND NECESSARY IN ORDER TO ADAPT IT TO THE USE OF THE AMERICAN STUDENT.

Second edition, revised and corrected.

CAMBRIDGE, N. E.

PRINTED BY HILLIARD AND METCALF, AT THE UNIVERSITY PRESS.

Sold by W. Hilliard, Cambridge, and by Cummings & Hilliard, No. 1 Cornhill, Boston.

1821.

ADVERTISEMENT.

The first principles, as well as the more difficult parts of Mathematics, have, it is thought, been more fully and clearly explained by the French elementary writers, than by the English; and among these, Lacroix has held a very distinguished place. His treatises have been considered as the most complete, and the best suited to those who are destined for a public education. They have received the sanction of the government, and have been adopted in the principal schools, of France. The following translation is from the thirteenth Paris edition. The original being written with reference to the new system of weights and measures, in which the different denominations proceed in a decimal ratio, it was found necessary to make considerable alterations and additions, to adapt it to the measures in use in the United States. The several articles relating to the reduction, addition, subtraction, multiplication, and division of compound numbers, have been written anew; a change has been made in many of the examples and questions, and new ones have been introduced after most of the rules, as an exercise for the learner.

JOHN FARRAR,
Professor of Mathematics and Natural Philosophy in the University at Cambridge.

Cambridge, Aug. 1818.

CONTENTS.

	Page.
General remarks on the different kinds of magnitude or quantity	1
Of number - - - - - - - - -	ibid.
Of spoken numeration - - - - - - -	2
Manner of representing numbers by figures, or written numeration - - - - - - -	4
Of reading numbers - - - - -	6
Of abstract and concrete numbers - - - -	7

Of Addition.

Of the principles on which addition is founded - -	7
General rule for performing addition - - -	9

Of Subtraction.

Of the principles on which it is founded - - -	9
Explanation of the terms, *remainder*, *excess*, and *difference* -	10
General rule for subtraction - - - -	12
Method of proof for addition and subtraction - -	13

Of Multiplication.

The origin of multiplication - - - -	14
An explanation of the terms *multiplicand*, *multiplier*, *product*, and *factors* - - - - - - -	15
Of the principles on which multiplication is performed -	ibid.
The table of Pythagoras, containing the products of any two figures - - - - - -	16
Formation of this table - - - - - -	ibid.
Remarks, from which it is inferred, that a change in the order of the factors does not affect the product - -	17

Rule for multiplying a number, consisting of several figures, by a single figure - - - - - - 18
To multiply by 10, 100, 1000, &c. - - - - 19
Rule for multiplying by a number consisting of a single digit and any number of ciphers - - - - 20
General rule for multiplication - - - - 21
Manner of abridging the process, when both factors are terminated by ciphers - - - - - - 22

Of Division.

The origin of division - - - - - 22
Explanation of the terms, *dividend, divisor,* and *quotient* - 23
Of the principles on which division is founded - - ibid.
Mode of proceeding, when the divisor consists of several figures 27
General rule for division - - - - - 28
Method of abbreviating the process of division - - 29
When both the divisor and dividend have ciphers on the right 30
Multiplication and division mutually prove each other - ibid.

Of Fractions.

The origin of fractions - - - - - 30
The manner of reading and writing fractions - - 32
An explanation of the terms *numerator* and *denominator* ibid.
Of the changes which a fraction undergoes, by the increase or diminution of one of its terms - - - 33
A Table representing the changes which take place in a fraction, by the multiplication or division of either of its terms - 34
The value of a fraction not altered by multiplying or dividing both its terms by the same number - - - ibid.
To simplify a fraction without altering its value - - - 35
The greatest common divisor of two numbers - - 36
General rule for finding the greatest common divisor - - 38
To distinguish the numbers divisible by 2, 5, or 3 - - 39
Of prime numbers - - - - - 40
General signification of the term multiplier - - - 41
To multiply a whole number by a fraction - - 42
To find the whole number contained in a fraction - - 43
To reduce a whole number to a fraction - - - 44

To multiply one fraction by another - - - 44
Of compound fractions - - - - - 45
Of division in general - - - - - ibid.
Of the division of a whole number by a fraction - - 46
To divide one fraction by another - - - ibid.
Of the addition and subtraction of fractions - - - 47
To reduce fractions to a common denominator - - ibid.
Of the addition and subtraction of mixed numbers - - 49
The product of several factors not changed, by changing the order in which they are multiplied - - - 50

Of Decimal Fractions.

The origin of decimal fractions - - - - 51
The manner of reading and writing decimals - - 52
A number containing decimals, not altered by annexing ciphers 54
Addition of decimals - - - - - ibid.
Subtraction of decimals - - - - - 55
The effect of changing the place of the decimal point - - 56
To multiply a number containing decimals, by a whole number 57
The multiplication of one decimal by another - - 58
To divide a decimal number by a whole number - - 59
To divide one decimal by another - - - - ibid.
Method of approximating the quotient of a division by decimals ibid.
Note.—Method of finding the value of the quotient of a division in fractions of a given denomination - - 60
To reduce vulgar fractions to decimals - - - 61
Note.—On the changing of one fraction to another of a lower denomination - - - - - 62
Of periodical decimals - - - - - 63

Tables of Coin, Weight, and Measure.

Of Federal Money - - - - - 65
English Money - - - - - - 66
Troy Weight - - - - - ibid.
Apothecaries Weight - - - - - ibid.
Avoirdupois Weight - - - - - 67
Dry Measure - - - - - - ibid.
Ale and Beer Measure - - - - ibid.

Wine Measure - - - - - - 68
Cloth Measure - - - - - - ibid.
Long Measure - - - - - ibid.
Time - - - - - - - ibid.

Reduction.

To reduce pounds and shillings to pence &c. - - - 69
To reduce shillings to pounds - - - - 70
To reduce other denominations of money, weight, and measure 71
To reduce a compound number to the lowest denomination contained in it - - - - - - 72
To reduce a number from a lower denomination to a higher - ibid.
To reduce the several parts of a compound number to a fraction of the highest denomination contained in it - - 73
To find the value of a fraction of a higher denomination, in the terms of a lower - - - - - ibid.
To reduce the several parts of a compound number to a decimal of the highest denomination contained in it - - 75
To reduce a decimal of a higher denomination to a lower - 76
To convert shillings, pence, and farthings, to the decimal of a pound - - - - - - 77
To convert the decimal of a pound to shillings &c. - - ibid.
To reduce numbers from one denomination to another, when the two numbers are not commensurable - - 78

Of Compound Numbers.

General rule for the addition of compound numbers - - 80
Method of proving the addition of compound numbers - 81
Of the subtraction of compound numbers - - - 82
To prove subtraction of compound numbers - - 83
Multiplication of compound numbers - - - 84
General rule for the multiplication of compound numbers - 85
Mode of proceeding, when the multiplier exceeds 12 - - ibid.
Of duodecimals - - - - - - - - 86
General rule for the multiplication of duodecimals - - 87
Of the division of compound numbers - - - 89
General rule for compound division - - - 90
Method of operation, when the divisor is large, and resolvable into two or more factors - - - - ibid.

Process, when the divisor cannot be so resolved - - 91
Multiplication and division mutually prove each other - ibid.

Of Proportion.

A development of the principles on which the rules of proportion are founded - - - - - - 92
Of the nature of ratiors - - - - - 93
Explanation of the terms, *relation* and *ratio* - - ibid.
Of the term *proportion* - - - - - 94
Of the terms *antecedent* and *consequent* - - ibid.
Of the equality of the product of the means to that of the extremes 95
Transposition of the terms of a proportion - - - ibid.
To obtain any one term of the proportion from the other three 96
The *Rule of Three* - - - - - 97
Rules for the stating of questions - - - - ibid.
An elucidation of these rules - - - - 98
General rule for solving all questions in proportion - - 99
Examples for illustration - - - - - ibid.
Questions for practice - - - - - 100

Compound Proportion.

The *Double Rule of Three* - - - - 103
Of the principles on which it is founded - - - ibid.
These principles illustrated by examples - - 104
Of the compounding of ratios - - - - 106
General rule for solving questions in compound proportion - 107
Examples for practice - - - - - ibid.

Fellowship.

The use of the rule, and the principles on which it is founded 108
Commercial use of the terms, *capital* or *stock* and *dividend* - 109
Examples illustrative of the principles of fellowship - - ibid.
Of *equidifference* in numbers, or arithmetical ratios - 110
Note.—Distinction between *geometrical* and *arithmetical* proportion - - - - - - ibid.
Manner of writing numbers in *equidifference* - - 111
Questions for practice - - - - - ibid.

Of Alligation.

The principles of medial alligation explained - - 113
Illustrated by examples - - - - - ibid.
Alligation alternate explained - - - - 114
Examples for illustration - - - - - 115

Miscellaneous Questions - - 117

Appendix.

Tables of various weights and measures - - - 119
New French weights and measures - - - - ibid.
Reasons for adopting the decimal gradation - - ibid.
The measures of length - - - - - ibid.
The measures of capacity - - - - 120
Weights - - - - - - - ibid.
Land Measure - - - - - - 121
The division of the circle - - - - - ibid.
The decimal system of coin - - - - ibid.
Divisions of time - - - - - - ibid.
Scripture long measure - - - - - 122
Grecian long measure reduced to English - - ibid.
Jewish long or itinerary measure - - - - 123
Roman long measure reduced to English - - ibid.
Attic dry measure reduced to English - - - 124
Attic measures of capacity for liquids reduced to English wine measure - - - - - - ibid.
Measures of capacity for liquids reduced to English wine measure - - - - - - ibid.
Jewish dry measure reduced to English - - - 125
Jewish measure of capacity for liquids reduced to English wine measure - - - - - - ibid.
Roman dry measure reduced to English - - - ibid.
Of the principal gold and silver coins, containing their weights, fineness, pure contents, current value, &c. - - 126

Explanation of the Roman Numerals.

One	I
Two	II*
Three	III
Four	IV†
Five	V
Six	VI‡
Seven	VII
Eight	VIII
Nine	IX
Ten	X
Twenty	XX
Thirty	XXX
Forty	XL
Fifty	L
Sixty	LX
Seventy	LXX
Eighty	LXXX
Ninety	XC
Hundred	C
Two hundred	CC
Three hundred	CCC
Four hundred	CCCC

* As often as any character is repeated, so many times its value is repeated.

† A less character before a greater diminishes its value.

‡ A less character after a greater increases its value.

Five hundred	D or IƆ*
Six hundred	DC
Seven hundred	DCC
Eight hundred	DCCC
Nine hundred	DCCCC
Thousand	M or CIƆ†
Eleven hundred	MC
Twelve hundred	MCC
Thirteen hundred	MCCC
Fourteen hundred	MCCCC
Fifteen hundred	MD
Two thousand	MM
Five thousand	IƆƆ : or $\overline{V}$†
Six thousand	$\overline{VI}$
Ten thousand	$\overline{X}$ or CCIƆƆ
Fifty thousand	IƆƆƆ
Sixty thousand	$\overline{LX}$
Hundred thousand	$\overline{C}$ or CCCIƆƆƆ
Million	$\overline{M}$ or CCCCIƆƆƆƆ
Two millions	$\overline{MM}$
&c. &c.	

* For every Ɔ affixed this becomes ten times as many.

† For every C and Ɔ put one at each end, it is increased ten times.

‡ A line over any number increases it 1000 fold.

ELEMENTARY TREATISE

ON

ARITHMETIC.

NUMERATION.

1. A comparison of the different objects, that come within the reach of our senses, soon leads us to perceive, that, in all these objects, there is an attribute, or quality, by which they can be supposed susceptible of increase or diminution; this attribute is *magnitude*. It generally appears in two different forms. Sometimes as a collection of several similar things, or separate parts, and is then designated by the word *number*.

Sometimes it presents itself as a whole, without distinction of parts; it is thus, that we consider the distance between two points, or the length of a line extending from one to the other, as also the outlines and surfaces of bodies, which determine their figure and *extent*, and finally this *extent* itself.

The proper characteristic of this last kind of magnitude is the connexion or union of the parts, or their *continuity*; whilst in number we consider how many parts there are; a circumstance to which the word *quantity* at first had relation, though afterwards it was applied to magnitude in general, magnitude considered as a whole being called *continued quantity*, to distinguish it from number, which is called *discrete*, or *discontinued quantity*.

2. All that relates to magnitude is the object of *mathematics*; numbers, in particular, are the object of *arithmetic*.

Continued magnitude belongs to *geometry*, which treats of the properties presented by the forms of bodies, considered with regard to their *extent*.

3. Number, being a collection of many similar things, or many

distinct parts, supposes the existence of one of these things, or parts, taken as a term of comparison, and this is called *unity*.

The most natural mode of forming numbers is, to begin with joining one unity to another, then, to this sum another ; and continuing in this manner, we obtain collections of units, which are expressed by particular names ; the whole of these names, which varies in different languages, composes the *spoken numeration*.

4. As there are no limits to the extention of numbers, since however great a number may be, it is always possible to add an unit to it, we may easily conceive that there is an infinity of different numbers, and, consequently, that it would be impossible to express them in any language whatever, by names, that should be independent of each other.

Hence have arisen nomenclatures, in which the object has been, by the combinations of a small number of words, subject to regular forms, and therefore easily remembered, to give a great number of distinct expressions.

Those, which are in use in the [English language,] with few exceptions, are derived from the names assigned to the nine first numbers and those afterwards given to the collections of *ten*, a *hundred*, and a *thousand units*.

The units are expressed by

one, two, three, four, five, six, seven, eight, nine.

The collections of ten units, or *tens*, by

ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety.

The collections of ten tens, or *hundreds*, are expressed by names borrowed from the units ; thus we say,

hundred, two hundred, three hundred, ... nine hundred.

The collections of ten hundreds, or *thousands*, receive their denominations from the nine first numbers and from the collections of tens and hundreds ; thus we say

thousand, two thousand . . . . nine thousand,

ten thousand, twenty thousand, &c.

hundred thousand, two hundred thousand, &c.

The collections of ten hundred thousands, or of thousands of thousands, take the name of *millions*, and are distinguished, like the collections of thousands.

The collections of ten hundreds of millions, or of thousands of millions, are called *billions,* and are distinguished, like the collections of millions.†

---

† The idea of number is the latest and most difficult to form. Before the mind can arrive at such an abstract conception, it must be familiar with that process of classification, by which we successively remount from individuals to species, from species to genera, and from genera to orders. The savage is lost in his attempts at numeration, and significantly expresses his inability to proceed by holding up his expanded fingers, or pointing to the hairs of his head.

Nature has furnished the great and universal standard for computation in the fingers of the hand. All nations have accordingly reckoned by *fives ;* and some barbarous tribes have scarcely advanced any further. After the fingers of one hand had been counted once, it was a second and perhaps a distant step to proceed to those of the other. The primitive words, expressing numbers, did not probably exceed five. To denote *six, seven, eight,* and *nine*, the North American Indians repeat the five with the successive addition of one, two, three, and four ; could we safely trace the descent and affinity of the abbreviated terms denoting the numbers from five to ten, it seems highly probable, that we should discover a similar process to have taken place in the formation of the most refined languages.

The ten digits of both hands being reckoned up, it then became necessary to repeat the operation. Such is the foundation of our decimal scale of arithmetic. Language still betrays by its structure the original mode of proceeding. To express the numbers beyond ten, the Laplanders combine an ordinal with a cardinal digit. Thus, eleven, twelve, &c. they denominate *second* ten and one, *second* ten and two, &c. and in like manner they call twenty one, twenty two, &c. *third* ten and one, *third* ten and two, &c. Our term *eleven* is supposed to be derived from *ein* or *one*, and *liben*, *to remain*, and to signify *one*, *leave* or *set aside ten. Twelve* is of the like derivation, and means *two, laying aside the ten.* The same idea is suggested by our termination *ty* in the words *twenty, thirty*, &c. This syllable, altogether distinct from *ten*, is derived from *ziehen*, to *draw,* and the meaning of *twenty* is, strickly speaking, *two drawings*, that is, the hands have been twice closed and the fingers counted over.

After ten was firmly established, as the standard of numeration, it seemed the most easy and consistent to proceed by the same repeated

Each of the names just mentioned is considered as forming a unit of an order more elevated according as it is removed from simple unit. The names *ten* and *hundred* are continually repeated, and we have no occasion for new names, such as *thousand, million, billion,* except at every fourth order. The same law being observed, to billions succeed *trillions, quadrillions, quintillions,* &c. each, like billions, having its tens and hundreds.

Numbers expressed in this manner, when more than one word enters into the enunciation of them, are separated into their respective orders of units, mentioned above; for instance, the number expressed by *five hundred thousand three hundred and two,* is separated into three parts, viz. *five hundreds of thousands, three hundreds of simple units, and two of these units.*

5. The length of the expression, written in words, when the numbers were large, occasioned the invention of characters, exclusively adapted to a shorter representation, and hence originated the art of expressing numbers in writing by these characters, called *figures,* or *written numeration.*

The laws of the written numeration, now used, are very analogous to those of the spoken numeration. In it the nine first numbers are each represented by a particular character, viz.

1	2	3	4	5	6	7	8	9
one,	two,	three,	four,	five,	six,	seven,	eight,	nine.

When a number consists of tens and units, the characters representing the number of each are written in order from left to right, beginning with the tens. The number forty-seven, for instance, is written 47; the first figure on the left, 4, denotes the four tens, and consequently a value ten times greater than it would have standing alone; while the figure 7, placed on the right, expressing seven units, possesses only its original value.

---

composition. Both hands being closed ten times would carry the reckoning up to a *hundred.* This word, originally *hund,* is of uncertain derivation; but the term *thousand,* which occurs at the next stage of the progress, or the hundred added ten times, is clearly traced out, being only a contraction of *duis hund,* or *twice hundred,* that is, the *repetition,* or *collection of hundreds.* See Edinburgh Review, vol. XVIII. art. VII.

In the number thirty-three, which is written 33, we see the figure 3 repeated, but each time with a different value; the value of the 3 on the left is ten times greater than the value of that on the right.

This is the fundamental law of our written numeration, that *a removal, of one place, towards the left increases the value of a figure ten times.*

If it were required to express fifty, or five tens, as there are no units in this number, there would be nothing to write but the figure 5, and consequently it would be necessary to show, by some particular mark, that in the expression of this number, the figure ought to occupy the first place on the left. To do this we place on the right the character 0, *cipher* or *nought*, which of itself has no value, and serves only to fill the place of the units, which are wanting in the enunciation of the proposed number.

6. Thus with ten characters, by means of the rule before laid down concerning the value which figures assume, according to the places they occupy, we can express all possible numbers.

With two figures only, we can write all, as far as to nine tens and nine units, making 99, or ninety nine. After this comes the hundred, which is expressed by the figure 1, put one place farther towards the left, than it would be, if used to express tens only; and to denote this place, two ciphers are placed on the right, making 100.

The units and tens, afterwards added to form numbers greater than 100, take their proper places; thus a hundred and one will be written in figures 101; a hundred and eleven, 111. Here the same figure is three times repeated, and with a different value each time; in the first place on the right it expresses an unit, in the second, a ten, in the third, a hundred. It is the same with the number 222, 333, 444, &c. Thus, in consequence of the rule laid down before when speaking of units and tens, *the same figure expresses units ten times greater, in proportion as it is removed from right to left, and by a simple change of place, acquires the power of representing, successively, all the different collections of units, which can enter into the expression of a number.*

7. A number dictated, or enunciated, is written then, by placing one after the other, beginning at the left, the figures which express the number of units of each collection; but it is necessary to keep in mind the order in which the collections succeed each other, that no one may be omitted, and to put ciphers in the room of those, which are wanting in the enunciation of the number to be written. If, for example, the number were *three hundred and twenty-four thousand, nine hundred and four,* we should put 3 for the hundreds of thousands, 2 for the twenty thousand, or the two tens of thousands, 4 for the thousands, 9 for the hundreds; and as the tens come immediately after the hundreds, and are wanting in the given number, we should put a cipher in the room of them, and then write the figure 4 for the units; we should thus have 324904.

In the same way, writing ciphers in the place of tens of thousands, thousands and tens, which are wanting in the number five hundred thousand three hundred and two, we should have 500302.

8. When a number is written in figures, in enunciating it, or expressing it in language, it is necessary to substitute for each of the figures the word which it represents, and then to mention the collection of units, to which it belongs according to the place it occupies. The following example will illustrate this;

2	4,	8	9	7,	3	2	1,	5	8	0,	3	4	6,
Tens of Trillions	TRILLIONS	Hundreds of Billions	Tens of Billions	BILLIONS	Hundreds of Millions	Tens of Millions	MILLIONS	Hundreds of Thousands	Tens of Thousands	THOUSANDS	Hundreds	Tens	UNITS

The figures of this number are divided by commas, into portions of three figures each, beginning at the right; but the last division on the left, which in the present instance has but two figures, may sometimes have but one. Each of these divisions corresponds to the collections designated by the words *unit, thousand,* *million, billion, trillion,* and their figures express successively the units, tens, and hundreds of each. *Consequently, the expression of the whole number given is made in words, by reading each division of figures as if it stood alone, and adding, after its units, the name of their place.*

The above example is read, *twenty four trillions, eight hundred and ninety seven billions, three hundred and twenty one millions, five hundred and eighty thousand, three hundred and forty six units.*

9. Numbers admit of being considered in two ways; one is, when no particular denomination is mentioned, to which their units belong, and they are then called *abstract numbers*; the other when the denomination of their units is specified, as when we say, two men, five years, three hours, &c. these are called *concrete numbers.*

It is evident, that the formation of numbers, by the successive union of units, is independent of the nature of these units, and that this must also be the case with the properties resulting from this formation; by which properties we are enabled to compound and decompound numbers, which is called *calculation.* We shall now explain the principal rules for the calculation of numbers, without regard to the nature of their units.

ADDITION.

10. This operation, which has for its object the uniting of several numbers in one, is only an abbreviation of the formation of numbers by the successive union of units. If, for instance, it were required to add five to seven, it would be necessary, in the series of the names of numbers, *one, two, three, four, five, six, seven,* &c. to ascend five places above seven, and we should then come to the word *twelve,* which is consequently the amount of seven units added to five. It is upon this process that the addition of all small numbers depends, the results of which are committed to memory; its immediate application to larger numbers would be impossible, but in this case, we suppose these numbers divided into the different collections of units contained in them, and we may add together those of the same name. For instance, to add 27 to 32, we add the 7 units of the first number

to the 2 of the second, making 9; then the 2 tens of the first with the 3 of the second, making 5 tens. The two results, taken together, form a total of 5 tens and 9 units, or 59, which is the sum of the numbers proposed.

What is here said applies to all numbers, however large, that are to be added together; but it is necessary to observe, that the partial sums, resulting from the addition of two numbers, each expressed by a single figure, often contain tens, or units of the next higher collection, and these ought consequently to be joined to their proper collection.

In the addition of the numbers 49 and 78, the sum of the units 9 and 8 is 17, of which we should reserve 10, or ten, to be added to the sum of the tens in the given numbers; next we say that 4 and 7 make 11, and joining to this the ten we reserved, we have 12 for the number of tens contained in the sum of the given numbers; which sum, therefore, contains 1 hundred, 2 tens and 7 units, that is, 127.

11. By proceeding on these principles, a method has been devised of placing numbers, that are to be added, which facilitates the uniting of their collections of units, and a rule has been formed, which the following example will illustrate.

Let the numbers be 527, 2519, 9812, 73 and 8; in order to add them together, we begin by writing them under each other, placing the units of the same order in the same column; then we draw a line to separate them from the result, which is to be written underneath it.

$$\begin{array}{rr} & 527 \\ & 2519 \\ & 9812 \\ & 73 \\ & 8 \\ \hline \text{Sum} & 12939 \end{array}$$

We at first find the sum of the numbers contained in the column of units to be 29, we write down only the nine units, and reserve the 2 tens, to be joined to those which are contained in the next column, which, thus increased, contains 13 units of its own order; we write down here only the three units, and carry the ten to the next column. Proceeding with this column as with the others, we find its sum to be 19; we write down the 9 units and carry the ten to the next column, the sum of which we then find to be 12; we write down the 2 units under this column and place the ten on the left of it; that is, we write down the sum of this column, as it is found.

By this means we obtain 12939 for the sum of the given numbers.

12. The rule for performing this operation may be given thus, *Write the numbers to be added under each other, so that all the units of the same kind may stand in the same column, and draw a line under them.*

Beginning at the right, add up successively the numbers in each column; if the sum does not exceed 9, write it beneath its column, as it is found; if it contains one or more tens, carry them to the next column; lastly, under the last column write the whole of its sum†.

Examples for practice.

Add together 8635, 2194, 7421, 5063, 2196 and 1225. *Ans.* 26734.

Add together 84371, 6250, 10, 3842 and 631. *Ans.* 95104.

Add together 3004, 523, 8710, 6345 and 784. *Ans.* 19366.

Add together 7861, 345, 8023. *Ans.* 16229.

Add together 66947, 46742 and 132684. *Ans.* 246373.

SUBTRACTION.

13. After having learned to compose a number by the addition of several others, the first question, that presents itself, is, how to take one number from another that is greater, or which amounts to the same thing, to separate this last into two parts, one of which shall be the given number. If, for instance, we have the

† The best method of proving addition is by means of subtraction. The learner may, however, in general, satisfy himself of the correctness of his work by beginning at the top of each column and adding down, or by separating the upper line of figures and adding up the rest and then adding this sum to the upper line.

number 9, and we wish to take 4 from it, we should, by doing this, separate it into two parts, which by addition would be the same again.

To take one number from another, when they are not large, it is necessary to pursue a course opposite to that prescribed in the beginning of article 10, for finding their sum; that is, in the series of the names of numbers, we ought to begin from the greatest of the numbers in question, and descend as many places as there are units in the smallest, and we shall come to the name given to the difference required. Thus, in descending four places below the number *nine* we come to *five*, which expresses the number that must be added to 4 to make 9, or which shows how much 9 is greater than 4.

In this last point of view, 5 is the *excess* of 9 above 4. If we only wished to show the inequality of the numbers 9 and 4, without fixing our attention on the order of their values, we should say that their *difference* was 5. Lastly, if we were to go through the operation of taking 4 from 9, we should say that the *remainder* is 5. Thus we see that, although the words, *excess*, *remainder*, and *difference*, are synonymous, each answers to a particular manner of considering the separation of the number 9 into the parts 4 and 5, which operation is always designated by the name *subtraction*.

14. When the numbers are large, the subtraction is performed, part at a time, by taking successively from the units of each order in the greatest number, the corresponding units in the least. That this may be done conveniently, the numbers are placed as 9587 and 345 in the following example;

$$\begin{array}{lr} & 9587 \\ & 345 \\ \hline \text{Remainder} & 9242 \end{array}$$

and under each column is placed the excess of the upper number, in that column, over the lower, thus;

5, taken from 7, leaves 2,
4, taken from 8, leaves 4,
3, taken from 5, leaves 2,

and writing afterwards the figure 9, from which there is nothing to be taken; the remainder, 9242, shows how much 9587 is greater than 345.

That the process here pursued gives a true result is indisputable, because in taking from the greatest of the two numbers all the parts of the least, we evidently take from it the whole of the least.

15. The application of this process requires particular attention, when some of the orders of units in the upper number are greater than the corresponding orders in the lower.

If, for instance, 397 is to be taken from 524.

	524
	397
Remainder	127

In performing this question we cannot at first take the units in the lower number from those in the upper; but the number 524, here represented by 4 units, 2 tens and 5 hundreds, can be expressed in a different manner by decomposing some of its collections of units, and uniting a part with the units of a lower order. Instead of the 2 tens and 4 units which terminate it, we can substitute in our minds 1 ten and 14 units, then taking from these units the 7 of the lower number, we get the remainder 7. By this decomposition, the upper number now has but one ten, from which we cannot take the 9 of the lower number, but from the 5 hundred of the upper number we can take 1, to join with the ten that is left, and we shall then have 4 hundreds and 11 tens, taking from these tens the tens of the lower number, 2 will remain. Lastly, taking from the 4 hundreds, that are left in the upper number, the three hundreds of the lower, we obtain the remainder 1, and thus get 127 as the result of the operation.

This manner of working consists, as we see, in borrowing, from the next higher order, an unit, and joining it according to its value to those of the order, on which we are employed, observing to count the upper figure of the order from which it was borrowed one unit less, when we shall have come to it.

16. When any orders of units are wanting in the upper number, that is, when there are ciphers between its figures, it is

necessary to go to the first figure on the left, to borrow the 10 that is wanted. See an example

$$\begin{array}{r} 7002 \\ 3495 \\ \hline \end{array}$$

Remainder 3507.

As we cannot take the 5 units of the lower number from the 2 of the upper, we borrow 10 units from the 7000, denoted by the figure 7, which leaves 6990; joining the 10 we borrowed to the figure 2, the upper number is now decompounded into 6990 and 12; taking from 12 the 5 units of the lower number, we obtain 7 for the units of the remainder.

This first operation has left in the upper number 6990 units or 699 tens instead of the 700, expressed by the three last figures on the left; thus the places of the two ciphers are occupied by 9s, and the significant figure on the left is diminished by unity. Continuing the subtraction in the other columns in the same manner, no difficulty occurs, and we find the remainder, as put down in the example.

17. Recapitulating the remarks made in the two preceding articles, the rule to be observed in performing subtraction may be given thus. *Place the less number under the greater, so that their units of the same order may be in the same column, and draw a line under them; beginning at the right, take successively each figure of the lower number from the one in the same column of the upper; if this cannot be done, increase the upper figure by ten units, counting the next significant figure, in the upper number, less by unity, and if ciphers come between, regard them as* 9s.

18. For greater convenience, when it is necessary to decrease the upper figure by unity, we can suffer it to retain its value, and add this unit to the corresponding lower figure, which, thus increased, gives, as is wanted, a result one less than would arise from the written figures. In the first of the following examples, after having taken 6 units from 14, we count the next figure of the lower number 8, as 9, and so in the others.

Examples.

16844	10378	103034	49812002
9786	2437	69845	18924983
7058		33189	

173425	8037142	2123724	39742107
57632	5067310	1123467	25378421

Method of proving Addition and Subtraction.

19. In performing an operation, according to a process, the correctness of which is established upon fixed principles, we may nevertheless sometimes commit errors in the partial additions and subtractions, the results of which we seek in the memory. To prevent any mistake of this kind, we have recourse to a method, the reverse of the first operation, by which we ascertain whether the results are right; this is called *proving* the operation.

The proof of addition consists in subtracting successively from the sum of the numbers added, all the parts of these numbers, and if the work has been correctly performed, there will be no remainder. We will now show by the example given in article 11, how to perform all these subtractions at once.

```
      527
     2519
     9812
       73
        8
     ----
Sum 12939
     1120
     ----
```

We first add the numbers in the left hand column, which here contains thousands, and subtract the sum 11 from 12, which begins the preceding result, and write underneath the difference 1, produced by what was reserved from the column of hundreds, in performing the addition. The sum of the column of hundreds, taken by itself, amounts to but 18; if we take this from the 9 of the first result, increased by borrowing the one thousand, considered as ten hundred, that remains from the column preceding it on the left, the remainder 1, written beneath, will show what was reserved from the column of tens. The sum of the last 11, taken from 13, leaves for its remainder 2 tens, the number reserved from the column of units. Joining these 2 tens with the 9 units of the answer, we form the number 29, which ought to be exactly the sum of the column of units, as this column is not affected by any of the others; adding again the numbers in this column, we ought to come to the same result, and consequently to have no remainder. This is actually the case, as is denoted by the 0 written under the column. The process, just explained, may be given thus; *to prove addition. beginning on the left, add again each of the several columns, subtract the sums respectively from the sums written above them and write down the remainders, which must be joined, each as so many tens to the sum of the next column on the right; if the work be correct there will be no remainder under the last column.*

20. The proof of subtraction is, that *the remainder, added to the least number, exactly gives the greatest.* Thus to ascertain the exactness of the following subtraction,

$$\begin{array}{r} 524 \\ 297 \\ \hline 227 \\ \hline 524 \end{array}$$

we add the remainder to the smallest number, and find the sum, in reality, equal to the greatest.

MULTIPLICATION.

21. When the numbers to be added are equal to each other, addition takes the name of *multiplication*, because in this case the sum is composed of one of the numbers repeated as many times as there are numbers to be added. Reciprocally, if we wish to repeat a number several times, we may do it, by adding the number to itself as many times, wanting one, as it is to be repeated. For instance, by the following addition,

$$\begin{array}{r} 16 \\ 16 \\ 16 \\ 16 \\ \hline 64 \end{array}$$

the number 16 is repeated four times, and added to itself three times.

To repeat a number twice is to *double* it ; 3 times, to *triple* it ; 4 times, to *quadruple* it, and so on.

22. Multiplication implies three numbers, namely, that, which is to be repeated, and which is called the *multiplicand* ; the number which shows how many times it is to be repeated, which is called the *multiplier* ; and lastly the result of the operation, which is called the *product*. The *multiplicand* and *multiplier*, considered as concurring to form the product, are called *factors* of the *product*. In the example given above, 16 is the *multiplicand*, 4 the *multiplier*, and 64 the product ; and we see that 4 and 16 are the *factors* of 64.

23. When the multiplicand and multiplier are large numbers, the formation of the product, by the repeated addition of the multiplicand, would be very tedious. In consequence of this, means have been sought of abridging it, by separating it into a certain number of partial operations, easily performed by memory. For instance, the number 16 would be repeated 4 times, by taking separately, the same number of times, the six units and the ten, that compose it. It is sufficient then to know the products arising from the multiplication of the units of each order in the multiplicand by the multiplier, when the multiplier consists of a single figure, and this amounts, for all cases that can occur, to finding the products of each one of the 9 first numbers by every other of these numbers.

24. These products are contained in the following table, attributed to Pythagoras.

TABLE OF PYTHAGORAS.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

25. To form this table, the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, are written first on the same line. Each one of these numbers is then added to itself and the sum written in the second line, which thus contains each number of the first doubled, or the product of each number by 2. Each number of the second line is then added to the number over it in the first, and their sums are written in the third line, which thus contains the triple of each number in the first, or their products by 3. By adding the numbers of the third line to those of the first, a fourth is formed, containing the quadruple of each number of the first, or their products by 4; and so on, to the ninth line, which contains the products of each number of the first line by 9.

It may not be amiss to remark, that the different products of any number whatever by the numbers 2, 3, 4, 5, &c. are called *multiples* of that number; thus 6, 9, 12, 15, &c. are multiples of 3.

26. When the formation of this table is well understood, the mode of using it may be easily conceived. If, for instance, the product of 7 by 5 were required; looking to the fifth line, which contains the different products of the 9 first numbers by 5, we should take the one directly under the 7, which is 35; the same

method should be pursued in every other instance, and *the product will always be found in the line of the multiplier and under the multiplicand.*

27. If we seek in the table of Pythagoras the product of 5 by 7, we shall find, as before, 35, although in this case 5 is the multiplicand, and 7 the multiplier. This remark is applicable to each product in the table, and *it is possible, in any multiplication, to reverse the order of the factors; that is, to make the multiplicand the multiplier, and the multiplier the multiplicand.*

As the table of Pythagoras contains but a limited number of products, it would not be sufficient to verify the above conclusion by this table; for a doubt might arise respecting it in the case of greater products, the number of which is unlimited; there is but one method independent of the particular value of the multiplicand and multiplier of showing that there is no exception to this remark. This is one well calculated for the purpose, as it gives a good illustration of the manner, in which the product of two numbers is formed. To make it more easily understood, we will apply it first to the factors 5 and 3.

If we write the figure 1, 5 times on one line, and place two similar lines underneath the first, in this manner,

1, 1, 1, 1, 1,
1, 1, 1, 1, 1,
1, 1, 1, 1, 1,

the whole number of 1s will consist of as many times 5 as there are lines, that is, 3 times 5; but, by the disposition of these lines, the figures are ranged in columns, containing 3 each. Counting them in this manner, we find as many times 3 units as there are columns, or 5 times 3 units, and as the product does not depend on the manner of counting, it follows that 3 times 5 and 5 times 3 give the same product. It is easy to extend this reasoning to any numbers, if we conceive each line to contain as many units as there are in the multiplicand, and the number of lines, placed one under the other, to be equal to the multiplier. In counting the product by lines, it arises from the multiplicand repeated as many times as there are units in the multiplier; but the assemblage of figures written presents as many columns as there are units in a line, and each column contains as many units as there are lines; if, then, we choose to count by columns, the number of lines, or the multiplier, will be repeated as many times as there are units in a line, that is, in the multiplicand. We may therefore, in finding the product of any two numbers, take either of them at pleasure, for the multiplier.

28. The reasoning, just given to prove the truth of the preceding proposition, is the demonstration of it, and it may be remarked, that the essential distinction of pure mathematics is, that no proposition, or process, is admitted, which is not the necessary consequence of the primary notions, on which it is founded, or the truth of which is not generally established by reasoning independent of particular examples, which can never constitute a proof, but serve only to facilitate the reader's understanding the reasoning, or the practice of the rules.

29. Knowing all the products given by the nine first numbers, combined with each other, we can, according to the remark in article 23, multiply any number by a number consisting of a single figure, by forming successively the product of each order of units in the multiplicand, by the multiplier; the work is as follows;

$$\begin{array}{r} 526 \\ 7 \\ \hline 3682 \end{array}$$

The product of the units of the multiplicand, 6, by the multiplier, 7, being 42, we write down only the 2 units, reserving the 4 tens to be joined with those, that will be found in the next higher place.

The product of the tens of the multiplicand, 2, by the multiplier, 7, is 14, and adding the 4 tens we reserved, we make it 18, of which number we write only the units, and reserve the ten for the next operation.

The product of the hundreds of the multiplicand, 5, by the multiplier, 7, is 35; when increased by the 1 we reserved, it becomes 36, the whole of which is written, because there are no more figures in the multiplicand.

30. This process may be given thus; *To multiply a number*

of several figures by a single figure, place the multiplier under the units of the multiplicand, and draw a line beneath, to separate them from the product. Beginning at the right, multiply successively, by the multiplier, the units of each order in the multiplicand, and write the whole product of each, when it does not exceed 9; but, if it contains tens, reserve them to be added to the next product. Continue thus to the last figure of the multiplicand, on the left, the whole result of which must be written down.

Examples. 243 by 6. *Ans.* 1458. 8943 by 9. *Ans.* 80487.

It is evident that, when the multiplicand is terminated by 0, the operation can commence only with its first significant figure; but to give the product its proper value, it is necessary to put, on the right of it, as many 0s as there are in the multiplicand. As for the 0s, which may occur between the figures of the multiplicand, they give no product, and a 0 must be written down when no number has been reserved from the preceding product, as is shown by the following examples:

956	8200	7012	80970
6	9	5	4
5736	73800	35060	323880

Multiply

730 by 3. *Ans.* 2190. 8104 by 4. *Ans.* 32416.
20508 by 5. *Ans.* 102540. 360500 by 6. *Ans.* 2163000.
297000 by 7. *Ans.* 2079000. 9097030 by 9. *Ans.* 81873270.

31. The most simple number, expressed by several figures, being 10, 100, 1000, &c. it seems necessary to inquire how we can multiply any number by one of these. Now if we recollect the principle mentioned in article 6, by which the same figure is increased in value 10 times, by every remove towards the left, we shall soon perceive, that to multiply any number by 10, we must make each of its orders of units ten times greater; that is, we must change its units into tens, its tens into hundreds, and so on, and that this is effected by placing a 0 on the right of the number proposed, because then all its significant figures will be advanced one place towards the left.

For the same reason, to multiply any number by 100, we should place two ciphers on the right; for, since it becomes ten times greater by the first cipher, the second will make it ten times greater still, and consequently it will be 10 times 10, or 100 times, greater than it was at first.

Continuing this reasoning, it will be perceived that, according to our system of numeration, a number is multiplied by 10, 100, 1000, &c. by writing on the right of the multiplicand as many ciphers as there are on the right of the unit in the multiplier.

32. When the significant figure of the multiplier differs from unity, as, for instance, when it is required to multiply by 30, or 300, or 3000, which are only 10 times 3, or 100 times 3, or 1000 times 3, &c. the operation is made to consist of two parts, we at first multiply by the significant figure, 3, according to the rule in article 30, and then multiply the product by 10, 100, or 1000, &c. (as was stated in the preceding article) by writing one, two, three, &c. ciphers on the right of this product.

Let it be required, for instance, to multiply 764 by 300.

$$\begin{array}{r} 764 \\ 300 \\ \hline 229200 \end{array}$$

The four significant figures of this product result from the multiplication of 764 by 3, and are placed two places towards the left to admit the two ciphers, which terminate the multiplier.

In general, *when the multiplier is terminated by a number of ciphers, first multiply the multiplicand by the significant figure of the multiplier, and place, after the product, as many ciphers as there are in the multiplier.*

Examples.

Multiply

35012 by 100. *Ans.* 3501200. 638427 by 500. *Ans.* 319213500.
2107900 by 70. *Ans.* 147553000. 9120400 by 90. *Ans.* 820836000.

33. The preceding rules apply to the case, in which the multiplier is any number whatever, by considering separately each of the collections of units of which it is composed. To multiply, for instance, 793 by 345, or, which is the same thing, to repeat 793, 345 times, is to take 793, 5 times, added to 40 times, added to

300 times, and the operation to be performed is resolved into 3 others, in each of which the multipliers, 5, 40, and 300, have but one significant figure.

To add the result of these three operations easily, the calculation is disposed thus ;

$$\begin{array}{r} 793 \\ 345 \\ \hline 3965 \\ 31720 \\ 237900 \\ \hline 273585 \end{array}$$

The multiplicand is multiplied successively by the units, tens, hundreds, &c. of the multiplier, observing to place a cipher on the right of the partial product, given by the tens in the multiplier, and two on the right of the product given by hundreds, which advances the first of these products one place towards the left, and the second, two. The three partial products are then added together, to obtain the total product of the given numbers.

As the ciphers, placed at the end of these partial products, are of no value in the addition, we may dispense with writing them, provided we take care to put in its proper place the first figure of the product given by each significant figure of the multiplier ; that is, to put in the place of tens the first figure of the product given by the tens in the multiplier ; in the place of hundreds the first figure of the product given by the hundreds in the multiplier, and so on.

34. According to what has been said, the rule is as follows. *To multiply any two numbers, one by the other, form successively (according to the rule in article 30,) the products of the multiplicand, by the different orders of units in the multiplier ; observing to place the first figure of each partial product under the units of the same order with the figure of the multiplier, by which the product is given ; and then add together all the partial products.*

35. When the multiplicand is terminated by ciphers, they may at first be neglected, and all the partial multiplications begin with the first significant figure of the multiplicand ; but after-

wards, to put in their proper rank the figures of the total product, as many ciphers, as there are in the multiplicand, must be written on the right of this product.

If the multiplier is terminated by ciphers, we may, according to the remark in article 31, neglect these also, provided we write an equal number on the right of the product.

Hence it results that, *when both multiplicand and multiplier are terminated by ciphers, these ciphers may at first be neglected, and after the other figures of the product are obtained, the same number may be written on the right of the product.*

When there are ciphers between the significant figures of the multiplier, as they give no product, they may be passed over, observing to put in its proper place the unit of the product, given by the figure on the left of these ciphers.

Examples.

300	526	Multiply 9648 by 5137. *Ans.* 49561776.
40	307	7854 by 350. *Ans.* 2748900.
———	———	17204774 by 125. *Ans.* 2150596750.
12000	3682	62500 by 520. *Ans.* 32500000.
	157800	25980762 by 40. *Ans.* 1039230480.
	———	
	161482	

DIVISION.

36. The product of two numbers being formed by repeating one of these numbers as many times as there are units in the other, we can, from the product, find one of the factors, by ascertaining how many times it contains the other; subtraction alone is necessary for this. Thus, if it be required to ascertain the number of times 64 contains 16, we need only subtract 16 from 64 as many times as it can be done; and since, after 4 subtractions, nothing is left, we conclude, that 16 is contained 4 times in 64. This manner of decomposing one number by another, in order to know how many times the last is contained in the first, is called *division*, because it serves to divide, or portion out, a given number into equal parts, of which the number or value is given.

If, for instance, it were required to divide 64 into 4 equal parts; to find the value of these parts, it would be necessary to ascertain the number, that is contained 4 times in 64, and consequently to regard 64 as a product, having for its factors 4 and one of the required parts, which is here 16.

If it were asked how many parts, of 16 each, 64 is composed of, it would be necessary, in order to ascertain the number of these parts, to find how many times 64 contains 16, and consequently, 64 must be regarded as a product, of which one of the factors is 16, and the other the number sought, which is 4.

Whatever then may be the object in view, *division consists in finding one of the factors of a given product, when the other is known.*

37. The number to be divided is called the *dividend*, the factor, that is known, and by which we must divide, is called the *divisor*, the factor found by the division is called the *quotient*, and always shows how many times the divisor is contained in the dividend.

It follows then, from what has been said, *that the divisor multiplied by the quotient ought to reproduce the dividend.*

38. When the dividend can contain the divisor a great many times, it would be inconvenient in practice to make use of repeated subtraction for finding the quotient; it then becomes necessary to have recourse to an abbreviation analogous to that which is given for multiplication. If the dividend is not ten times larger than the divisor. which may be easily perceived by the inspection of the numbers, and if the divisor consists of only one figure, the quotient may be found by the table of Pythagoras, since that contains all the products of factors that consist of only one figure each. If it were asked, for instance, how many times 8 is contained in 56, it would be necessary to go down the 8th column, to the line in which 56 is found; the figure 7, at the beginning of this line, shows the second factor of the number 56, or how many times 8 is contained in this number.

We see by the same table, that there are numbers, which cannot be exactly divided by others. For instance, as the seventh line, which contains all the multiples of 7, has not 40 in it, it follows that 40 is not divisible by 7; but as it comes between 35 and 42, we see that the greatest multiple of 7, it can contain, is 35, the factors of which are 5 and 7. By means of this elementary information, and the considerations, which will now be offered, any division whatever may be performed.

39. Let it be required, for example, to divide 1656 by 3; this question may be changed into another form, namely; *To find such a number, that multiplying its units, tens, hundreds, &c. by 3, the product of these units, tens, hundreds, &c. may be the dividend,* 1656.

It is plain, that this number will not have units of a higher order than thousands, for, if it had tens of thousands, there would be tens of thousands in the product, which is not the case. Neither can it have units of as high an order as thousands, for if it had but one of this order, the product would contain at least 3, which is not the case. It appears then, that the thousand in the dividend is a number reserved, when the hundreds of the quotient were multiplied by 3, the divisor.

This premised, the figure occupying the place of hundreds, in the required quotient, ought to be such, that, when multiplied by 3, its product may be 16, or the greatest multiple of 3 less than 16. This restriction is necessary, on account of the reserved numbers, which the other figures of the quotient may furnish, when multiplied by the divisor, and which should be united to the product of the hundreds.

The number, which fulfils this condition, is 5; but 5 hundreds, multiplied by 3, gives 15 hundreds, and the dividend, 1656, contains 16 hundreds; the difference, 1 hundred, must have come then from the reserved number, arising from the multiplication of the other figures of the quotient by the divisor. If we now subtract the partial product, 15 hundreds, or 1500, from the total product, 1656, the remainder, 156, will contain the product of the units and tens of the quotient by the divisor, and the question will be reduced to finding a number, which, multiplied by 3, gives 156, a question similar to that, which presented itself above. Thus when the first figure of the quotient shall have been found in this last question, as it was in the first, let it be multiplied by the divisor, then subtracting this partial product from the whole product, the result will be a new dividend, which may be treated in the same manner as the preceding, and so on, until the original dividend is exhausted.

40. The operation just described is disposed of thus;

$$
\begin{array}{lr|l}
\text{dividend} & 1656 & \underline{\;3\;}\ \text{divisor} \\
 & \underline{15} & 552\ \text{quotient} \\
 & 15 & \\
 & \underline{15} & \\
 & 06 & \\
 & \underline{6} & \\
 & 0 &
\end{array}
$$

The dividend and divisor are separated by a line, and another line is drawn under the divisor, to mark the place of the quotient. This being done, we take on the left of the dividend the part 16, capable of containing the divisor, 3, and dividing it by this number, we get 5 for the first figure of the quotient on the left; then taking the product of the divisor by the number just found. and subtracting it from 16, the partial dividend, we write, underneath, the remainder, 1, by the side of which we bring down the 5 tens of the dividend. Considering the number, as it now stands, a second partial dividend, we divide it also by the divisor, 3, and obtain 5 for the second figure of the quotient; we then take the product of this number by the divisor, and subtracting it from the partial dividend, get 0 for the remainder. We then bring down the last figure of the dividend, 6, and divide, this third partial dividend by the divisor, 3, and get 2 for the last figure of the quotient.

41. It is manifest that, if we find a partial dividend, which cannot contain the divisor, it must be because the quotient has no units of the order of that dividend, and that those which it contains arise from the products of the divisor by the units of the lower orders in the quotient; it is necessary, therefore, whenever this is the case, to put a 0 in the quotient, to occupy the place of the order of units that is wanting.

For instance, let 1535 be divided by 5.

1535	5
15	307
035	
35	
00	

The division of the 15 hundreds of the dividend, by the divisor, leaving no remainder, the 3 tens, which form the second partial dividend, do not contain the divisor. Hence it appears, that the quotient ought to have no tens; consequently this place must be filled with a cipher, in order to give to the first figure of the quotient the value, it ought to have, compared with the others; then bringing down the last figure of the dividend, we form a third partial dividend, which, divided by 5, gives 7 for the units of the quotient, the whole of which is now 307.

42. The considerations, presented in article 40, apply equally to the case, in which the divisor consists of any number of figures.

If, for instance, it were required to divide 57981 by 251, it would easily be seen, that the quotient can have no figures of a higher order than hundreds, because, if it had thousands, the dividend would contain hundreds of thousands, which is not the case; further, the number of hundreds should be such, that, multiplied by 251, the product would be 579, or the multiple of 251 next less than 579; this restriction is necessary on account of the reserved numbers which may have been furnished by the multiplication of the other figures of the quotient by the divisor. The number, which answers to this condition, is 2; but 2 hundreds, multiplied by 251, give 502 hundreds, and the divisor contains 579; the difference, 77 hundreds, arises from the reserved numbers resulting from the multiplication of the units and tens of the quotient, by the divisor.

If we now subtract the partial product, 502 hundreds, or 50200, from the total product, 57981, the remainder, 7781, will contain the products of the units and tens of the quotient by the divisor, and the operation will be reduced to finding a number, which, multiplied by 251, will give for a product 7781.

Thus, when the first figure of the quotient shall have been determined, it must be multiplied by the divisor, the product being subtracted from the whole dividend, a new dividend will be the result, which must be operated upon like the preceding ; and so on, till the whole dividend is exhausted.

It is always necessary, for obtaining the first figure of the quotient, to separate, on the left of the dividend, so many figures, as, considered as simple units, will contain the divisor, and admit of this partial division.

43. Disposing of the operation as before, the calculation, just explained, is performed in the following order ;

```
57981 | 251
502   |-----
----- | 231
 778
 753
 ---
  251
  251
  ---
  000
```

The 3 first figures, on the left of the dividend. are taken to form the partial dividend ; they are divided by the divisor, and the number 2, thence resulting, is written in the quotient ; the divisor is then multiplied by this number, and the product, 502, is written under the partial dividend, 579. Subtraction being performed, the 8 tens of the dividend are brought down to the side of the remainder, 77 ; this new partial dividend is then divided by the divisor, and 3 is obtained for the second figure of the quotient ; the divisor is multiplied by this, the product subtracted from the corresponding partial dividend, and to the remainder, 25, is brought down the last figure of the dividend, 1 ; this last partial dividend, 251, being equal to the divisor, gives 1 for the units of the quotient.

44. When the divisor contains many figures, some difficulty may be found in ascertaining how many times it is contained in

the partial dividends. The following example is designed to show how it may be known.

```
423405 | 485
3880   |----
------ | 873
 3540
 3395
 ----
  1455
  1455
  ----
  0000
```

It is necessary at first to take four figures on the left of the dividend, to form a number which will contain the divisor; and then it cannot be immediately perceived how many times 485 is contained in 4234. To aid us in this inquiry, we shall observe, that this divisor is between 400 and 500; and if it were exactly one or the other of these numbers, the question would be reduced to finding how many times 4 hundred or 5 hundred is contained in the 42 hundreds of the number 4234, or, which amounts to the same thing, how many times 4 or 5 is contained in 42. For the first of these numbers we get 10, and for the second 8; the quotient must now be sought between these two. We see at first that we cannot employ 10, because this would imply, that the order of units in the dividend above hundreds contained the divisor, which is not the case. It only remains then, to try which of the two numbers 9 or 8, used as the multiplier of 485, gives a product that can be subtracted from 4234, and 8 is found to be the one. Subtracting from the partial dividend the product of the divisor multiplied by 8, we get, for the remainder, 354; bringing down then the 0 tens in the dividend, we form a second partial dividend, on which we operate as on the preceding; and so with the others.

45. The recapitulation of the preceding articles gives us this rule, *To divide one number by another, place the divisor on the right of the dividend, separate them by a line, and draw another line under the divisor, to make the place for the quotient. Take, on the left of the dividend, as many figures as are necessary to contain the divisor; find how many times the number expressed by the first*

figure of the divisor, is contained in that, represented by the first, or two first, figures of the partial dividend; multiply this quotient, which is only an approximation, by the divisor, and, if the product is greater than the partial dividend, take units from the quotient continually, till it will give a product that can be subtracted from the partial dividend; subtract this product, and if the remainder be greater than the dividend, it will be a proof that the quotient has been too much diminished; and, consequently, it must be increased. By the side of the remainder bring down the next figure of the dividend, and find, as before, how many times this partial dividend contains the divisor; continue thus, until all the figures of the given dividend are brought down. When a partial dividend occurs, which does not contain the divisor, it is necessary, before bringing down another figure of the dividend, to put a cipher in the quotient.

46. The operations required in division may be made to occupy a less space, by performing mentally the subtraction of the products given by the divisor and each figure of the quotient, as is exhibited in the following example;

$$\begin{array}{r|l} 1755 & \underline{39} \\ 195 & 45 \\ 000 & \end{array}$$

After having found that the first partial dividend contains 4 times the divisor, 39, we multiply at first the 9 units by 4, which gives 36; and, in order to subtract this product from the partial dividend, we add to the 5 units in the dividend 4 tens, making their sum 45, from which taking 36, 9 remains. We then reserve 4 tens to join them, in the mind, to 12, the product of the quotient by the tens in the divisor, making the sum 16; in taking this sum from 17, we take away the 4 tens, with which we had augmented the units of the dividend, in order to perform the preceding subtraction. We then operate in the same manner on the second partial dividend, 195, saying; 9 times 5 make 45, taken from 45, nought remains, then 5 times 3 make 15, and 4 tens, reserved, make 19, taken from 19, nought remains.

We see sufficiently by this in what manner we are to perform any other example, however complicated.

47. Division is also abbreviated when the dividend and divi-

sor are terminated by ciphers, because we can strike out, from the end of each, as many ciphers as are contained in the one that has the least number.

If, for instance, 84000 were to be divided by 400, these numbers may be reduced to 840 and 4, and the quotient would not be altered; for we should only have to change the name of the units, since, instead of 84000, or 840 hundreds, and 400, or 4 hundreds, we should have 840 units and 4 units, and the quotient of the numbers 840 and 4 is always the same, whatever may be the denomination of their units.

It may also be remarked that, in striking out two ciphers at the end of the given numbers, they have been, at the same time, both of them divided by 100; for it follows from article 31, that in striking out 1, 2, or 3 ciphers on the right of any number, the number is divided by 10, or 100, or 1000, &c.

Examples in Division.

```
144 |  3      16512 | 344      3049164 | 6274
 24 |----      2752 |-----      53956  |------
 00 | 48       0000 |  48        37644 |  486
                                 00000
```

Divide	49561776	by 5137.	*Ans.* 9648.
	27489000	by 350.	*Ans.* 7854.
	2150596750	by 125.	*Ans.* 17204774.
	32500000	by 520.	*Ans.* 62500.
	1039230480	by 40.	*Ans.* 25980762.

48. Division and multiplication mutually prove each other, like subtraction and addition, for according to the definition of division, (36), we ought, by dividing the product by one of the factors, to find the other; and multiplying the divisor by the quotient, we ought to reproduce the dividend (37).

FRACTIONS.

49. Division cannot always be exactly performed, because any number whatever of units taken a certain number of times, does not always compose any other number whatever. Exam-

ples of this have already been seen in the table of Pythagoras, which contains only the product of the 9 first numbers multiplied two and two, but does not contain all the numbers between 1 and 81, the first and last numbers in it. The method hitherto given shows then, only how to find the greatest multiple of the divisor, that can be contained in the dividend.

If we divide 239 by 8, according to the rule in article 46,

$$\begin{array}{r|l} 239 & \underline{\ 8\ } \\ 79 & 29 \\ 7 & \end{array}$$

we have, for the last partial dividend, the number 79, which does not contain 8 exactly, but which, falling between the two numbers, 72 and 80, one of which contains the divisor, 8, nine times, and the other ten, shows us that the last part of the quotient is greater than 9, and less than 10, and consequently, that the whole quotient is between 29 and 30. If we multiply the unit figure of the quotient, 9, by the divisor, 8, and subtract the product from the last partial dividend, 79, the remainder, 7, will evidently be the excess of the dividend, 239, above the product of the factors, 29 and 8. Indeed, having, by the different parts of the operation, subtracted successively from the dividend, 239, the product of each figure of the quotient by the divisor, we have evidently subtracted the product of the whole quotient by the divisor, or 232; and the remainder, 7, less than the divisor, proves, that 232 is the greatest multiple of 8, that can be contained in 239.

50. It must be perceived, after what has been said, that to reproduce any dividend, we must add to the product of the divisor by the quotient, the sum which remains when the division cannot be performed exactly.

51. If we wished to divide into eight equal parts a sum of whatever nature, consisting of 239 units, we could not do it without using parts of units or *fractions*. Thus, when we have taken from the number 239 the 8 times 29 units contained in it, there will remain 7 units, to be divided into 8 parts; to do this, we may divide each of these units, one after the other, into 8 parts, and then take one part out of each unit, which will give 7 parts to be joined to the 29 whole units, to form the eighth part of 239, or the exact quotient of this number, by 8.

The same reasoning may be applied to every other example of division in which there is a remainder, and in this case the quotient is composed of two parts; one, consisting of whole units, while the other cannot be obtained until the concrete or material units of the remainder have been actually divided into the number of parts denoted by the divisor; without this it can only be indicated by supposing, *a unit of the dividend to be divided into as many parts as there are units in the divisor, and so many of these parts, as there are units in the remainder, taken to complete the quotient required.*

52. In general, when we have occasion to consider quantities less than unity, we suppose unity divided into a certain number of parts, sufficiently small to be contained a certain number of times in these quantities, or to *measure* them. In the idea thus formed of their magnitude there are two elements, namely, the number of times the measuring part is contained in unity, and the number of these parts found in the quantities.

A nomenclature has been made for fractions, which answers to this manner of conceiving and representing them.

That which results from the division of unity

into 2 parts is called	*a moiety* or *half,*
into 3 parts	*a third,*
into 4 parts	*a quarter* or *fourth,*
into 5 parts	*a fifth,*
into 6 parts	*a sixth,*

and so on, adding after the two first, the termination *th* to the number, which denotes how many parts are supposed to be in unity.

Every fraction then is expressed by two numbers; the first, which shows how many parts it is composed of, is called the *numerator,* and the other, which shows how many of these parts are necessary to form an unit, is called the *denominator,* because the denomination of the fraction is deduced from it. *Five sixths* of an unit is a fraction, the numerator of which is *five,* and the denominator *six.*

The *numerator* and the *denominator* together are called the *two terms* of the fraction.

Figures are used to shorten the expression of fractions, the denominator being written under the numerator, and separated from it by a line,

one third is written $\frac{1}{3}$,
five sixths $\frac{5}{6}$.

53. According to the meaning attached to the words, *numerator* and *denominator*, it is plain, *that a fraction is increased, by increasing its numerator, without changing its denominator;* for this last, as it shows into how many parts unity is divided, determines the magnitude of these parts, which continues the same, while the denominator remains unchanged; and by augmenting the numerator, the number of these parts is augmented, and consequently the fraction increased. It is thus, for instance, that $\frac{8}{9}$ exceeds $\frac{7}{9}$, and that $\frac{13}{30}$ exceeds $\frac{11}{30}$.

It follows evidently from this, *that by repeating the numerator 2, 3, or any number of times, without altering the denominator, we repeat, a like number of times, the quantity expressed by the fraction, or in other words multiply it by this number;* for we make 2, 3, or any number of times, as many parts, as it had before, and these parts have remained each of the same value. The fraction $\frac{3}{5}$, then, is the triple of $\frac{1}{5}$ and $\frac{10}{21}$ the double of $\frac{5}{21}$.

A fraction is diminished by diminishing its numerator, without changing its denominator, since it is made to consist of a less number of parts than it contained before, and these parts retain the same value. Whence. *if the numerator be divided by 2, 3, or any number, without the denominator being altered, the fraction is made a like number of times smaller, or is divided by that number,* for it is made to contain 2, 3, or any number of times less parts than it contained before, and these parts remain of the same value. Thus $\frac{1}{5}$ is a third of $\frac{3}{5}$ and $\frac{5}{21}$ is half of $\frac{10}{21}$.

54. On the contrary, a fraction is diminished, when its denominator is increased without changing its numerator; for then more parts are supposed in an unit, and consequently they must be smaller, but, as only the same number of them are taken to form the fraction, the amount in this case must be a less quantity than in the first. Thus $\frac{2}{5}$ is less than $\frac{2}{3}$, and $\frac{4}{13}$ than $\frac{4}{9}$.

Hence it follows, that *if the denominator of a fraction be multiplied by 2, 3, or any number, without the numerator being changed,* *the fraction becomes a like number of times smaller, or is divided by that number*, for it is composed of the same number of parts as before, but each of them has become 2, 3, or a certain number of times less. The fraction $\frac{3}{8}$ is half of $\frac{3}{4}$, and $\frac{4}{15}$ the third of $\frac{4}{5}$.

A fraction is increased when its denominator is diminished without the numerator being changed; because, as unity is supposed to be divided into fewer parts, each one becomes greater, and their amount is therefore greater.

Whence, *if the denominator of a fraction be divided by 2, 3, or any other number, the fraction will be made a like number of times greater, or will be multiplied by that number;* for the number of parts remains the same, and each one becomes 2, 3, or a certain number of times greater than it was before. According to this, $\frac{3}{6}$ is triple of $\frac{3}{18}$ and $\frac{5}{6}$ the quadruple of $\frac{5}{24}$.

It may be remarked, that to suppress the denominator of a fraction is the same as to multiply the fraction by that number. For instance, to suppress the denominator 3 in the fraction $\frac{2}{3}$ is to change it into 2 whole ones, or to multiply it by 3.

55. The preceding propositions may be recapitulated as follows;

$$\left.\begin{array}{l}\text{By multiplying}\\ \text{By dividing}\end{array}\right\}\text{ the numerator, the fraction is }\left\{\begin{array}{l}\text{multiplied.}\\ \text{divided.}\end{array}\right.$$

$$\left.\begin{array}{l}\text{By multiplying}\\ \text{By dividing}\end{array}\right\}\text{ the denominator, the fraction is }\left\{\begin{array}{l}\text{divided.}\\ \text{multiplied}\end{array}\right.$$

56. The first consequence to be drawn from this table is, that the operations performed on the denominator produce effects of an *inverse* or *contrary* nature with respect to the value of the fraction. Hence it results, *that, if both the numerator and denominator of a fraction be multiplied at the same time, by the same number, the value of the fraction will not be altered;* for if, on the one hand, multiplying the numerator makes the fraction 2, 3, &c. times greater, so on the other, by the second operation, the half or third part &c. of it is taken; in other words, it is divided by the same number, by which it had at first been multiplied. Thus $\frac{1}{5}$ is equal to $\frac{3}{15}$, and $\frac{5}{21}$ is equal to $\frac{10}{42}$.

57. It is also manifest that, *if both the numerator and denominator of a fraction be divided, at the same time, by the same number, the value of the fraction will not be altered;* for if, on the one hand, by dividing the numerator the fraction is made 2, 3, &c.

times smaller ; on the other, by the second operation, the double, triple, &c. is taken ; in short it is multiplied by the same number, by which it was at first divided. Thus the fraction $\frac{2}{4}$ is equal to $\frac{1}{2}$, and $\frac{3}{9}$ is equal to $\frac{1}{3}$.

58. It is not with fractions as with whole numbers, in which a magnitude, so long as it is considered with relation to the same unit, is susceptible of but one expression. In fractions on the contrary, the same magnitude can be expressed in an infinite number of ways. For instance, the fractions

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \text{\&c.}$$

in each of which the denominator is twice as great as the numerator, express, under different forms, the half of an unit. The fractions $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \frac{6}{18}, \frac{7}{21}$, &c.

of which the denominator is three times as great as the numerator, represent each the third part of an unit. Among all the forms, which the given fraction assumes, in each instance, the first is the most remarkable, as being the most simple ; and, consequently, it is well to know how to find it from any of the others. It is obtained by dividing the two terms of the others by the same number, which, as has already been shown, does not alter their value. Thus if we divide by 7 the two terms of the fraction $\frac{7}{14}$, we come back to $\frac{1}{2}$; and, performing the same operation on $\frac{7}{21}$, we get $\frac{1}{3}$.

59. It is by following this process, that a fraction is reduced to its most *simple terms ;* it cannot, however, be applied, except to fractions, of which the numerator and denominator are divisible by the same number ; in all other cases the given fraction is the most simple of all those, that can represent the quantity it expresses. Thus the fractions $\frac{5}{7}$, $\frac{7}{12}$, $\frac{15}{16}$, the terms of which cannot be divided by the same number, or *have no common divisor,* are *irreducible,* and, consequently, cannot express, in a more simple manner, the magnitudes which they represent.

60. Hence it follows, that to simplify a fraction, we must endeavour to divide its two terms by some one of the numbers, 2, 3, &c ; but by this uncertain mode of proceeding it will not be always possible to come at the most *simple terms* of the given fraction, or at least, it will often be necessary to perform a great number of operations.

If, for instance, the fraction $\frac{24}{84}$ were given, it may be seen at once, that each of its terms is a multiple of 2, and dividing them by this number, we obtain $\frac{12}{42}$; dividing these last also by 2, we obtain $\frac{6}{21}$. Although much more simple now than at first, this fraction is still susceptible of reduction, for its two terms can be divided by 3; and it then becomes $\frac{2}{7}$.

If we observe, that to divide a number by 2, then the quotient by 2, and then the second quotient by 3, is the same thing as to divide the original number by the product of the numbers, 2, 2, and 3, which amounts to 12, we shall see that the three above operations can be performed at once by dividing the two terms of the given fraction by 12, and we shall again have $\frac{2}{7}$.

The numbers 2, 3, 4, and 12, each dividing the two numbers 24 and 84 at the same time, are the common divisors of these numbers; but 12 is the most worthy of attention, because it is the greatest, and it is by employing *the greatest common divisor* of the two terms of the given fraction, that it is reduced at once to its most simple terms. We have then this important problem to solve, *two numbers being given, to find their greatest common divisor*†.

61. We arrive at the knowledge of the common divisor of two numbers by a sort of trial easily made, and which has this recommendation, that each step brings us nearer and nearer to the number sought. To explain it clearly, I will take an example.

Let the two numbers be 637 and 143. It is plain, that the greatest common divisor of these two numbers cannot exceed the smallest of them; it is proper then to try if the number 143, which divides itself and gives 1 for the quotient, will also divide the number 637, in which case it will be the greatest common divisor sought. In the given example this is not the case; we obtain a quotient 4, and a remainder 65.

Now it is plain, that every common divisor of the two numbers, 143 and 637, ought also to divide 65, the remainder resulting from their division; for the greater, 637, is equal to the

---

† What is here called the *greatest common divisor*, is sometimes called the *greatest common measure*.

less, 143, multiplied by 4, plus the remainder, 65, (50) ; now in dividing 637 by the common divisor sought, we shall have an exact quotient ; it follows then, that we must obtain a like quotient. by dividing the assemblage of parts, of which 637 is composed, by the same divisor ; but the product of 143 by 4 must necessarily be divisible by the common divisor, which is a factor of 143, and consequently the other part, 65, must also be divisible by the same divisor ; otherwise the quotient would be a whole number accompanied by a fraction, and consequently could not be equal to the whole number, resulting from the division of 637 by the common divisor. By the same reasoning, it may be proved in general, *that every common divisor of two numbers must also divide the remainder resulting from the division of the greater of the two by the less.*

According to this principle, we see, that the common divisor of the numbers 637 and 143, must also be the common divisor of the numbers 143 and 65 ; but as the last cannot be divided by a number greater than itself, it is necessary to try 65 first. Dividing 143 by 65, we find a quotient 2, and a remainder 13 ; 65 then is not the divisor sought. By a course of reasoning, similar to that pursued with regard to the numbers, 637, 143, and the remainder, resulting from their division, 65, it will be seen that every common divisor of 143 and 65 must also divide the numbers 65 and 13 ; now the greatest common divisor of these two last cannot exceed 13 ; we must therefore try, if 13 will divide 65, which is the case, and the quotient is 5 ; then 13 is the greatest common divisor sought.

We can make ourselves certain of its possessing this property by resuming the operations in an inverse order, as follows ;

As 13 divides 65 and 13, it will divide 143, which consists of twice 65 added to 13 ; as it divides 65 and 143, it will divide 637, which consists of 4 times 143 added to 65 ; 13 then is the common divisor of the two given numbers. It is also evident, by the very mode of finding it, that there can be no common divisor greater than 13, since 13 must be divided by it.

It is convenient in practice, to place the successive divisions one after the other, and to dispose of the operation, as may be seen in the following example ;

637		143		65		13
572	4	130	2	65	5	
65		13		0		

the quotients, 4, 2, 5, being separated from the other figures.

The reasoning, employed in the preceding example, may be applied to any numbers, and thus conduct us to this general rule. *The greatest common divisor of two numbers will be found, by dividing the greater by the less; then the less by the remainder of the first division; then this remainder, by the remainder of the second division; then this second remainder by the third, or that of the third division; and so on, till we arrive at an exact quotient; the last divisor will be the common divisor sought.*

62. See two examples of the operation.

9024		3760		1504		752
7520	2	3008	2	1504	2	
1504		752		00		

752 then is the greatest common divisor of 9024 and 3760.

937		47		44		3		2		1
47	19	44	1	3	4	2	1	2	2	
467		3		14		1		0		
423				12						
44				2						

By this last operation we see that the greatest common divisor of 937 and 47, is 1 only, that is, these two numbers, properly speaking, have no common divisor, since all whole numbers, like them, are divisible by 1.

We may easily satisfy ourselves, that the rule of the preceding article must necessarily lead to this result, whenever the given numbers have no common divisor; for the remainders, each being less than the corresponding divisor, become less and less every operation, and it is plain, that the division will continue as long as there is a divisor greater than unity.

63. After these calculations, the fraction $\frac{143}{637}$ and $\frac{3760}{9024}$, can be at once reduced to their most simple term, by dividing the terms of the first by their common divisor, 13, and the terms of the second, by their common divisor, 752; we thus obtain $\frac{11}{49}$

and $\frac{5}{12}$. As to the fraction, $\frac{47}{937}$, it is altogether irreducible, since its terms have no common divisor but unity.

64. It is not always necessary to find the greatest common divisor of the given fraction; there are, as has before been remarked, reductions, which present themselves without this preparatory step.

Every number terminated by one of the figures 0, 2, 4, 6, 8, is necessarily divisible by 2; for in dividing any number by 2, only 1 can remain from the tens; the last partial division can be performed on the numbers 0, 2, 4, 6, 8, if the tens leave no remainder, and on the numbers 10, 12, 14, 16, 18, if they do, and all these numbers are divisible by 2.

The numbers divisible be 2 are called *even numbers*, because they can be divided into two equal parts.

Also, every number terminated on the right by a cipher, or by 5, is divisible by 5, for when the division of the tens by 5 has been performed, the remainder, if there be one, must necessarily be either 1, 2, 3, or 4, the remaining part of the operation will be performed on the numbers 0, 5, 10, 15, 20, 25, 30, 35, 40, or 45, all of which are divisible by 5.

The numbers, 10, 100, 1000, &c. expressed by unity followed by a number of ciphers, can be resolved into 9 added to 1, 99 added to 1, 999 added to 1, and so on; and the numbers 9, 99, 999, &c. being divisible by 3, and by 9, it follows that, if numbers of the form 10, 100, 1000, &c. be divided by 3 or 9, the remainder of the division will be 1.

Now every number which, like 20, 300, or 5000, is expressed by a single significant figure, followed on the right by a number of ciphers, can be resolved into several numbers expressed by unity, followed on the right by a number of ciphers; 20 is equal to 10 added to 10; 300, to 100 added to 100 added to 100; 5000, to 1000 added to 1000 added to 1000 added to 1000 added to 1000; and so with others. Hence it follows, that if 20, or 10 added to 10, be divided by 3 or 9, the remainder will be 1 added to 1, or 2; if 300, or 100 added to 100 added to 100, be divided by 3 or 9, the remainder will be 1 added to 1 added to 1, or 3.

In general, if we resolve in the same manner a number expressed by one significant figure, followed, on the right, by a number of ciphers, in order to divide it by 3 or 9; the remainder of this division will be equal to as many times 1, as there are units in the significant figure, that is, it will be equal to the significant figure itself. Now any number being resolved into units, tens, hundreds, &c. is formed by the union of several numbers expressed by a single significant figure; and, if each of these last be divided by 3 or 9, the remainder will be equal to one of the significant figures of the given number; for instance, the division of hundreds will give, for a remainder, the figure occupying the place of hundreds; that of tens, the figure occupying the place of tens; and so of the others. If then, the sum of all these remainders be divisible by 3 or 9, the division of the given number by 3 or 9 can be performed exactly; whence it follows, that if the sum of the figures, constituting any number, be divisible by 3 or 9, the number itself is divisible by 3 or 9.

Thus the numbers, 423, 4251, 15342, are divisible by 3, because the sum of the significant figures is 9 in the first, 12 in the second, and 15 in the third.

Also, 621, 8280, 934218, are divisible by 9, because the sum of the significant figures is 9 in the first, 18 in the second, and 27 in the third.

It must be observed, that every number divisible by 9 is also divisible by 3, although every number divisible by 3 is not also divisible by 9.

Observations might be made on several other numbers analogous to those just given on 2, 3, 5, and 9; but this would lead me too far from the subject.

The numbers 1, 3, 5, 7, 11, 13, 17, &c. which can be divided only by themselves, and by unity, are called *prime numbers;* two numbers, as 12 and 35, having, each of them, divisors, but neither of them any one, that is common to it with the other, are called *prime to each other.*

Consequently, the numerator and denominator of an irreducible fraction are prime to each other.

Examples for practice under Article 61.

What is the greatest common divisor of 24 and 36? *Ans.* 12.

What is the greatest common divisor of 35 and 100? *Ans.* 5.

What is the greatest common divisor of 312 and 504? *Ans.* 24.

Examples for practice under articles 57, 58, *and* 60.

Reduce $\frac{25}{75}$ to its most simple terms. *Ans.* $\frac{1}{3}$.
Reduce $\frac{512}{4096}$ to its most simple terms. *Ans.* $\frac{1}{8}$.
Reduce $\frac{81}{729}$ to its most simple terms. *Ans.* $\frac{1}{9}$.
Reduce $\frac{160}{168}$ to its most simple terms. *Ans.* $\frac{20}{21}$.
Reduce $\frac{324}{378}$ to its most simple terms. *Ans.* $\frac{6}{7}$.
Reduce $\frac{2640}{2880}$ to its most simple terms. *Ans.* $\frac{11}{12}$.

65. After this digression we will resume the examination of the table in article 55,

$$\left.\begin{array}{l}\text{By multiplying}\\ \text{By dividing}\end{array}\right\}\text{ the numerator, the fraction is }\left\{\begin{array}{l}\text{multiplied,}\\ \text{divided,}\end{array}\right.$$
$$\left.\begin{array}{l}\text{By multiplying}\\ \text{By dividing}\end{array}\right\}\text{ the denominator, the fraction is }\left\{\begin{array}{l}\text{divided,}\\ \text{multiplied,}\end{array}\right.$$

that we may deduce from it some new inferences.

We see at once, by an inspection of this table, that a fraction can be multiplied in two ways, namely, by multiplying its numerator, or dividing its denominator, and that it can also be divided in two ways, namely, by dividing its numerator, or multiplying its denominator; hence it follows, that multiplication alone, according as it is performed on the numerator or denominator, is sufficient for the multiplication and division of fractions by whole numbers. Thus $\frac{3}{15}$, multiplied by 7 units, makes $\frac{21}{15}$; $\frac{4}{9}$, divided by 3, makes $\frac{4}{27}$.

Examples for practice.

Multiply $\frac{2}{3}$ by 5. *Ans.* $\frac{10}{3}$.
Multiply $\frac{4}{21}$ by 4. *Ans.* $\frac{16}{21}$.
Multiply $\frac{3}{48}$ by 6. *Ans.* $\frac{3}{8}$.
Multiply $\frac{5}{9}$ by 30. *Ans.* $\frac{150}{9}$.
Multiply $\frac{1}{30}$ by 5. *Ans.* $\frac{1}{6}$.
Multiply $\frac{2}{45}$ by 9. *Ans.* $\frac{2}{5}$.

Divide $\frac{3}{6}$ by 3. *Ans.* $\frac{1}{6}$.
Divide $\frac{4}{18}$ by 6. *Ans.* $\frac{1}{27}$.
Divide $\frac{5}{6}$ by 10. *Ans.* $\frac{1}{12}$.
Divide $\frac{7}{9}$ by 8. *Ans.* $\frac{7}{72}$.
Divide $\frac{20}{25}$ by 4. *Ans.* $\frac{1}{5}$.
Divide $\frac{22}{11}$ by 4. *Ans.* $\frac{1}{2}$.

66. The doctrine of fractions enables us to generalize the definition of multiplication given in article 21. When the multi-

plier is a whole number, it shows how many times the multiplicand is to be repeated ; but the term multiplication, extended to fractional expressions, does not always imply augmentation, as in the case of whole numbers. To comprehend in one statement every possible case, it may be said, *that to multiply one number by another is, to form a number by means of the first, in the same manner as the second is formed, by means of unity.* In reality, when it is required to multiply by 2, by 3, &c. the product consists of twice, three times, &c. the multiplicand, in the same way as the multiplier consists of two, three, &c. units ; and to multiply any number by a fraction, $\frac{1}{5}$ for example, is to take the fifth part of it, because the multiplier $\frac{1}{5}$, being the fifth part of unity, shows that the product ought to be the fifth part of the multiplicand*.

Also, to multiply any number by $\frac{4}{5}$ is to take out of this number or the multiplicand, a part, which shall be four fifths of it, or equal to four times one fifth.

Hence it follows, *that the object in multiplying by a fraction, whatever may be the multiplicand, is, to take out of the multiplicand a part, denoted by the multiplying fraction ;* and that this operation is composed of two others, namely, a division and a multiplication, in which the divisor and multiplier are whole numbers.

Thus, for instance, to take $\frac{4}{5}$ of any number, it is first necessary to find the fifth part, by dividing the number by 5, and to repeat this fifth part four times, by multiplying it by 4.

We see, in general, *that the multiplicand must be divided by the denominator of the multiplying fraction, and the quotient be multiplied by its numerator.*

The multiplier being less than unity, the product will be smaller than the multiplicand, to which it would be only equal, if the multiplier were 1.

67. If the multiplicand be a whole number divisible by 5, for

---

* We are led to this statement, by a question which often presents itself; namely, where the price of any quantity of a thing is required, the price of the unity of the thing being known. The question evidently remains the same, whether the given quantity be greater or less than this unity.

instance, 35, the fifth part will be 7; this result, multiplied by 4, will give 28 for the $\frac{4}{5}$ of 35, or for the product of 35 by $\frac{4}{5}$. If the multiplicand, always a whole number, be not exactly divisible by 5, as, for instance, if it were 32, the division by 5 will give for a quotient $6\frac{2}{5}$; this quotient repeated 4 times will give $24\frac{8}{5}$.

This result presents a fraction in which the numerator exceeds the denominator, but this may be easily explained. The expression $\frac{8}{5}$, in reality denoting 8 parts, of which 5, taken together, make unity, it follows, that $\frac{8}{5}$ is equivalent to unity added to three fifths of unity, or $1\frac{3}{5}$; adding this part to the 24 units, we have $25\frac{3}{5}$ for the value of $\frac{4}{5}$ of 32.

68. It is evident, from the preceding example, that the fraction $\frac{8}{5}$ contains unity, or *a whole one*, and $\frac{3}{5}$, and the reasoning, which led to this conclusion, shows also, that every fractional expression, of which the numerator exceeds the denominator, contains one or more units, or whole ones, and *that these whole ones may be extracted by dividing the numerator by the denominator; the quotient is the number of units contained in the fraction, and the remainder, written as a fraction, is that, which must accompany the whole ones.*

The expression $\frac{307}{53}$, for instance, denoting 307 parts, of which 53 make unity, there are, in the quantity represented by this expression, as many whole ones, as the number of times 53 is contained in 307; if the division be performed, we shall obtain 5 for the quotient, and 42 for the remainder; these 42 are fifty third parts of unity; thus, instead of $\frac{307}{53}$, may be written $5\frac{42}{53}$.

Examples for practice.

Reduce the fraction $\frac{6}{3}$ to its equivalent whole number.

Ans. 2.

Reduce $\frac{7}{2}$ to its equivalent whole or mixed number. *Ans.* $3\frac{1}{2}$.

Reduce $\frac{15}{4}$ to its equivalent whole or mixed number.

Ans. $3\frac{3}{4}$.

Reduce $\frac{482}{20}$ to its equivalent whole or mixed number.

Ans. $24\frac{2}{20}$.

Reduce $\frac{97}{8}$ to its equivalent whole or mixed number.

Ans. $12\frac{1}{8}$.

Reduce $\frac{512}{50}$ to its equivalent whole or mixed number.

Ans. $10\frac{6}{25}$.

69. The expression $5\frac{42}{53}$, in which the whole number is given, being composed of two different parts, we have often occasion to convert it into the original expression $\frac{307}{53}$, which is called, *reducing a whole number to a fraction.*

To do this, the *whole number is to be multiplied by the denominator of the accompanying fraction, the numerator to be added to the product, and the denominator of the same fraction to be given to the sum.*

In this case, the 5 whole ones must be converted into fifty-thirds, which is done by multiplying 53 by 5, because each unit must contain 53 parts; the result will be $\frac{265}{53}$; joining this part with the second, $\frac{42}{53}$, the answer will be $\frac{307}{53}$.

Examples for practice.

Reduce $12\frac{1}{2}$ to a fraction. *Ans.* $\frac{25}{2}$.
Reduce $6\frac{5}{9}$ to a fraction. *Ans.* $\frac{59}{9}$.
Reduce $31\frac{7}{10}$ to a fraction. *Ans.* $\frac{317}{10}$.
Reduce $45\frac{21}{130}$ to a fraction. *Ans.* $\frac{5871}{130}$.

70. We now proceed to the multiplication of one fraction by another.

If, for instance, $\frac{2}{3}$ were to be multiplied by $\frac{4}{5}$; according to article 66, the operation would consist in dividing $\frac{2}{3}$ by 5, and multiplying the result by 4; according to the table in article 65, the first operation is performed by multiplying 3, the denominator of the multiplicand, by 5; and the second, by multiplying 2, the numerator of the multiplicand, by 4; and the required product is thus found to be $\frac{8}{15}$.

It will be the same with every other example, and it must consequently be concluded from what precedes, *that to obtain the product of two fractions, the two numerators must be multiplied, one by the other, and under the product must be placed the product of the denominators.*

Examples.

Multiply $\frac{1}{5}$ by $\frac{3}{4}$. *Ans.* $\frac{3}{20}$. Multiply $\frac{4}{9}$ by $\frac{2}{7}$. *Ans.* $\frac{8}{63}$.

Multiply $\frac{5}{6}$ by $\frac{3}{8}$. *Ans.* $\frac{5}{16}$. Multiply $\frac{25}{30}$ by $\frac{10}{12}$. *Ans.* $\frac{25}{36}$.
Multiply $\frac{7}{21}$ by $\frac{1}{2}$. *Ans.* $\frac{1}{6}$. Multiply $\frac{11}{13}$ by $\frac{31}{42}$. *Ans.* $\frac{341}{546}$.

71. It may sometimes happen that two mixed numbers, or whole numbers joined with fractions, are to be multiplied, one by the other, as, for instance, $3\frac{5}{7}$ by $4\frac{8}{9}$. The most simple mode of obtaining the product is, to reduce the whole numbers to fractions by the procees in article 69 ; the two factors will then be expressed by $\frac{26}{7}$ and $\frac{44}{9}$, and their product, by $\frac{1144}{63}$ or $18\frac{10}{63}$, by extracting the whole ones (68).

72. The name *fractions of fractions* is sometimes given to the product of several fractions ; in this sense we say, $\frac{2}{3}$ of $\frac{4}{5}$. This expression denotes $\frac{2}{3}$ of the quantity represented by $\frac{4}{5}$ of the original unit, and taken in its stead for unity. These two fractions are reduced to one by multiplication (70), and the result, $\frac{8}{15}$, expresses the value of the quantity required, with relation to the original unit; that is, $\frac{2}{3}$ of the quantity represented by $\frac{4}{5}$ of unity is equivalent to $\frac{8}{15}$ of unity. If it were required to take $\frac{7}{9}$ of this result, it would amount to taking $\frac{7}{9}$ of $\frac{2}{3}$ of $\frac{4}{5}$, and these fractions, reduced to one, would give $\frac{56}{135}$ for the value of the quantity sought, with relation to the original unit.

73. The word *contain*, in its strict sense, is not more proper in the different cases presented by division, than the word *repeat* in those presented by multiplication ; for it cannot be said that the dividend contains the divisor, when it is less than the latter ; the expression is generally used, but only by analogy and extension.

To generalize division, *the dividend must be considered as having the same relation to the quotient, that the divisor has to unity,* because the divisor and quotient are the two factors of the dividend (36). This consideration is conformable to every case that division can present. When, for instance, the divisor is 5, the dividend is equal to 5 times the quotient, and, consequently, this last is the fifth part of the dividend. If the divisor be a fraction, $\frac{1}{2}$ for instance, the dividend cannot be but half of the quotient, or the latter must be double the former.

The definition, just given, easily suggests the mode of proceeding, when the divisor is a fraction. Let us take, for example, $\frac{4}{5}$. In this case the dividend ought to be only $\frac{4}{5}$ of the quotient; but $\frac{1}{5}$ being $\frac{1}{4}$ of $\frac{4}{5}$, we shall have $\frac{1}{5}$ of the quotient, by taking $\frac{1}{4}$ of the dividend, or dividing it by 4. Thus knowing $\frac{1}{5}$ of the quotient, we have only to take it 5 times, or multiply it by 5, to obtain the quotient. In this operation the dividend is divided by 4 and multiplied by 5, which is the same as taking $\frac{5}{4}$ of the dividend, or multiplying it by $\frac{5}{4}$, which fraction is no other than the divisor inverted.

This example shows, that, in general, *to divide any number by a fraction, it must be multiplied by the fraction inverted.*

For instance, let it be required to divide 9 by $\frac{3}{4}$; this will be done by multiplying it by $\frac{4}{3}$, and the quotient will be found to be $\frac{36}{3}$ or 12. Also 13 divided by $\frac{5}{7}$ will be the same as 13 multiplied by $\frac{7}{5}$ or $\frac{91}{5}$. The required quotient will be $18\frac{1}{5}$, by extracting the whole ones (68).

It is evident that, whenever the numerator of the divisor is less than the denominator, the quotient will exceed the dividend, because the divisor in that case, being less than unity, must be contained in the dividend a greater number of times, than unity is, which, taken for a divisor, always gives a quotient exactly the same as the dividend.

74. *When the dividend is a fraction, the operation must be performed by multiplying the dividend by the divisor inverted* (70).

Let it be required to divide $\frac{7}{8}$ by $\frac{2}{3}$; according to the preceding article, $\frac{7}{8}$ must be multiplied by $\frac{3}{2}$, which gives $\frac{21}{16}$.

It is evident, that the above operation may be enunciated thus; *To divide one fraction by another, the numerator of the first must be multiplied by the denominator of the second, and the denominator of the first, by the numerator of the second.*

If there be whole numbers joined to the given fractions, they must be reduced to fractions, and the above rule applied to the results.

Examples.

Divide 9 by $\frac{2}{5}$. *Ans.* $\frac{45}{2}$.
Divide 18 by $\frac{6}{5}$. *Ans.* 15.
Divide $\frac{3}{6}$ by $\frac{7}{9}$. *Ans.* $\frac{9}{14}$.
Divide $\frac{10}{11}$ by $\frac{4}{30}$. *Ans.* $\frac{75}{11}$.
Divide $7\frac{1}{2}$ by $\frac{1}{3}$. *Ans.* $\frac{45}{2}$.
Divide $2\frac{2}{3}$ by $3\frac{1}{4}$. *Ans.* $\frac{32}{39}$.
Divide $\frac{63}{9}$ by $\frac{9}{63}$. *Ans.* 49.
Divide $\frac{44}{11}$ by $\frac{44}{11}$. *Ans.* 1.

75. It is important to observe, that any division, whether it can be performed in whole numbers or not, may be indicated by a fractional expression; $\frac{36}{3}$, for instance, expresses evidently the quotient of 36 by 3, as well as 12, for $\frac{1}{3}$ being contained three times in unity, $\frac{36}{3}$ will be contained 3 times in 36 units, as the quotient of 36 by 3 must be.

76. It may seem preposterous to treat of the multiplication and division of fractions before having said any thing of the manner of adding and subtracting them; but this order has been followed, because multiplication and division follow as the immediate consequences of the remark given in the table of article 55, but addition and subtraction require some previous preparation. It is, besides, by no means surprising, that it should be more easy to multiply and divide fractions, than to add and subtract them, since they are derived from division, which is so nearly related to multiplication. There will be many opportunities, in what follows, of becoming convinced of this truth; that operations to be performed on quantities are so much the more easy, as they approach nearer to the origin of these quantities. We will now proceed to the addition and subtraction of fractions.

77. When the fractions on which these operations are to be performed have the same denominator, as they contain none but parts of the same denomination, and consequently of the same magnitude or value, they can be added or subtracted in the same manner as whole numbers, care being taken to mark, in the result, the denomination of the parts, of which it is composed.

It is indeed very plain, that $\frac{2}{11}$ and $\frac{3}{11}$ make $\frac{5}{11}$, as 2 quantities and 3 quantities of the same kind make 5 of that kind, whatever it may be.

Also, the difference between $\frac{3}{9}$ and $\frac{8}{9}$ is $\frac{5}{9}$, as the difference between 3 quantities and 8 quantities, of the same kind, is 5 of that kind, whatever it may be. Hence it must be concluded, that, *to add or subtract fractions, having the same denominator, the sum or difference of their numerators must be taken, and the common denominator written under the result.*

78. When the given fractions have different denominators, it

is impossible to add together, or subtract, one from the other, the parts of which they are composed, because these parts are of different magnitudes; but to obviate this difficulty, the fractions are made to undergo a change, which brings them to parts of the same magnitude, by giving them a common denominator.

For instance, let the fractions be $\frac{2}{3}$ and $\frac{4}{5}$; if each term of the first be multiplied by 5, the denominator of the second, the first will be changed into $\frac{10}{15}$; and if each term of the second be multiplied by 3, the denominator of the first, the second will be changed into $\frac{12}{15}$; thus two new expressions will be formed, having the same value as the given fractions (56).

This operation, necessary for comparing the respective magnitudes of two fractions, consists simply in finding, to express them, parts of an unit sufficiently small to be contained exactly in each of those which form the given fractions. It is plain, in the above example, that the fifteenth part of an unit will exactly measure $\frac{1}{3}$ and $\frac{1}{5}$ of this unit, because $\frac{1}{3}$ contains five 15ths, and $\frac{1}{5}$ contains three 15ths. The process, applied to the fractions $\frac{2}{3}$ and $\frac{4}{5}$, will admit of being applied to any others.

In general, *to reduce any two fractions to the same denominator, the two terms of each of them must be multiplied by the denominator of the other.*

79. *Any number of fractions are reduced to a common denominator, by multiplying the two terms of each by the product of the denominators of all the others;* for it is plain that the new denominators are all the same, since each one is the product of all the original denominators, and that the new fractions have the same value as the former ones, since nothing has been done except multiplying each term of these by the same number (56).

Examples.

Reduce $\frac{3}{4}$ and $\frac{5}{9}$ to a common denominator. *Ans.* $\frac{27}{36}$, $\frac{20}{36}$.

Reduce $\frac{8}{10}$ and $\frac{3}{7}$ to a common denominator. *Ans.* $\frac{56}{70}$. $\frac{30}{70}$.

Reduce $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$ to a common denominator. *Ans.* $\frac{20}{60}$, $\frac{45}{60}$, $\frac{48}{60}$.

Reduce $\frac{2}{10}$, $\frac{3}{5}$, $\frac{4}{7}$, and $\frac{5}{9}$ to a common denominator.

Ans. $\frac{630}{3150}$, $\frac{1890}{3150}$, $\frac{1800}{3150}$, $\frac{1750}{3150}$.

The preceding rule conducts us, in all cases, to the proposed end; but when the denominators of the fractions in question are not prime to each other, there is a common denominator more simple than that which is thus obtained, and which may be shown to result from considerations analogous to those given in the preceding articles. If, for instance, the fractions were $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, as nothing more is required, for reducing them to a common denominator, than to divide unity into parts, which shall be exactly contained in those of which these fractions consist, it will be sufficient to find the smallest number, which can be exactly divided by each of their denominators, 3, 4, 6, 8; and this will be discovered by trying to divide the multiples of 3 by 4, 6, 8; which does not succeed until we come to 24, when we have only to change the given fractions into 24[ths] of an unit.

To perform this operation we must ascertain successively how many times the denominators, 3, 4, 6, and 8, are contained in 24, and the quotients will be the numbers, by which each term of the respective fractions must be multiplied, to be reduced to the common denominator, 24. It will thus be found, that each term of $\frac{2}{3}$ must be multiplied by 8, each term of $\frac{3}{4}$ by 6, each term of $\frac{5}{6}$ by 4, and each term of $\frac{7}{8}$ by 3; the fractions will then become $\frac{16}{24}$, $\frac{18}{24}$, $\frac{20}{24}$, $\frac{21}{24}$.

Algebra will furnish the means of facilitating the application of this process.

80. By reducing fractions to the same denominator, they may be added and subtracted as in article 77.

81. When there are at the same time both whole numbers and fractions, the whole numbers, if they stand alone, must be converted into fractions of the same denomination as those which are to be added to them, or subtracted from them; and if the whole numbers are accompanied with fractions, they must be reduced to the same denominator with these fractions.

It is thus, that the addition of four units and $\frac{2}{3}$ changes itself into the addition of $\frac{12}{3}$ and $\frac{2}{3}$, and gives for the result $\frac{14}{3}$.

To add $3\frac{2}{7}$ to $5\frac{4}{9}$, the whole numbers must be reduced to fractions, of the same denomination as those which accompany them, which reduction gives $\frac{23}{7}$ and $\frac{49}{9}$; with these results the sum is found to be $\frac{550}{63}$, or $8\frac{46}{63}$. If, lastly, $\frac{4}{7}$ were to be subtracted from

$3\frac{1}{4}$, the operation would be reduced to taking $\frac{4}{5}$ from $\frac{13}{4}$, and the remainder would be $\frac{49}{20}$.

Examples in addition of fractions.

Add $\frac{2}{3}$ to $\frac{3}{9}$. *Ans.* $\frac{27}{27}$, or 1.
Add $\frac{5}{7}$ to $\frac{15}{20}$. *Ans.* $\frac{41}{28}$.
Add $\frac{3}{7}$ to $\frac{5}{2}$. *Ans.* $\frac{41}{14}$.
Add $\frac{3}{7}$, $\frac{4}{2}$, and $\frac{3}{5}$ together. *Ans.* $3\frac{1}{35}$.
Add $2\frac{1}{2}$, $4\frac{3}{4}$, and $5\frac{1}{3}$ together. *Ans.* $12\frac{7}{12}$.
Add $\frac{4}{5}$, $1\frac{1}{5}$, and $6\frac{3}{5}$ together. *Ans.* $8\frac{3}{5}$.

Examples in subtraction of fractions.

From $\frac{2}{3}$ take $\frac{1}{3}$. *Ans.* $\frac{1}{3}$.
From $\frac{3}{4}$ take $\frac{5}{9}$. *Ans.* $\frac{7}{36}$.
From $\frac{13}{20}$ take $\frac{4}{10}$. *Ans.* $\frac{1}{4}$.
From $5\frac{3}{8}$ take $2\frac{1}{2}$. *Ans.* $2\frac{7}{8}$.
From $8\frac{2}{3}$ take $4\frac{1}{5}$. *Ans.* $4\frac{7}{15}$.
From $3\frac{7}{9}$ take $2\frac{10}{11}$. *Ans* $\frac{86}{99}$.

82. The rule given, for the reduction of fractions to a common denominator supposes, that a product resulting from the successive multiplication of several numbers into each other, does not vary, in whatever order these multiplications may be performed; this truth, though almost always considered as self-evident, needs to be proved.

We shall begin with showing, that to multiply one number by the product of two others, is the same thing as to multiply it at first by one of them, and then to multiply that product by the other. For instance, instead of multiplying 3 by 35, the product of 7 and 5, it will be the same thing if we multiply 3 by 5, and then that product by 7. The proposition will be evident, if, instead of 3, we take an unit; for 1, multiplied by 5, gives 5, and the product of 5 by 7 is 35, as well as the product of 1 by 35; but 3, or any other number, being only an assemblage of several units, the same property will belong to it, as to each of the units of which it consists; that is, the product of 3 by 5 and by 7, obtained in either way, being the triple of the result given by unity, when multiplied by 5 and 7, must necessarily be the same. It may be proved in the same manner, that were it required to multiply 3 by the product of 5, 7, and 9, it would consist in multiplying 3 by 5, then this product by 7, and the result by 9, and so on, whatever might be the number of factors.

To represent in a shorter manner several successive multiplications, as of the numbers 3, 5, and 7, into each other, we shall write 3 by 5 by 7.

This being laid down, in the product 3 by 5, the order of the factors, 3 and 5 (27), may be changed, and the same product obtained. Hence it directly follows, that 5 by 3 by 7 is the same as 3 by 5 by 7.

The order of the factors 3 and 7, in the product 5 by 3 by 7, may also be changed, because this product is equivalent to 5, multiplied by the product of the numbers 3 and 7; thus we have in the expression 5 by 7 by 3, the same product as the preceding.

By bringing together the three arrangements,

3 by 5 by 7
5 by 3 by 7
5 by 7 by 3,

we see that the factor 3 is found successively, the first, the second, and the third, and that the same may take place with respect to either of the others. From this example, in which the particular value of each number has not been considered, it must be evident, that a product of three factors does not vary, whatever may be the order in which they are multiplied.

If the question were concerning the product of four factors, such as 3 by 5 by 7 by 9, we might, according to what has been said, arrange, as we pleased, the three first or the three last, and thus make any one of the factors pass through all the places. Considering then one of the new arrangements, for instance this, 5 by 7 by 3 by 9, we might invert the order of the two last factors, which would give 5 by 7 by 9 by 3, and would put 3 in the last place. This reasoning may be extended without difficulty to any number of factors whatever.

DECIMAL FRACTIONS.

83. Although we can, by the preceding rules, apply to fractions, in all cases, the four fundamental operations of arithmetic, yet it must have been long since perceived, that, if the different subdivisions of a unit, employed for measuring quantities small-

er than this unit, had been subjected to a common law of decrease, the calculus of fractions would have been much more convenient, on account of the facility with which we might convert one into another. By making this law of decrease conform to the basis of our system of numeration, we have given to the calculus the greatest degree of simplicity, of which it is capable.

We have seen in article 5, that each of the collections of units contained in a number, is composed of ten units of the preceding order, as the ten consists of simple units ; but there is nothing to prevent our regarding this simple unit, as containing ten parts, of which each one shall be a *tenth ;* the tenth as containing ten parts, of which each one shall be a *hundredth* of unity, the hundredth as containing ten parts, of which each one shall be a *thousandth* of unity, and so on.

Proceeding thus, we may form quantities as small as we please, by means of which it will be possible to measure any quantities, however minute. These fractions, which are called *decimals,* because they are composed of parts of unity, that become continually ten times smaller, as they depart further from unity, may be converted, one into the other, in the same manner as *tens, hundreds, thousands,* &c. are converted into units ; thus,

the unit being equivalent to	10 tenths,
the tenth	10 hundredths,
the hundredth	10 thousandths,

it follows, that the tenth is equivalent to 10 times 10 thousandths, or 100 thousandths.

For instance, 2 tenths, 3 hundredths, and 4 thousandths will be equivalent to 234 thousandths, as 2 hundreds, 3 tens, and 4 units make 234 units ; and what is here said may be applied universally, since the subordination of the parts of unity is like that of the different orders of units.

84. According to this remark, we can, by means of figures, write decimal fractions in the same manner as whole numbers, since by the nature of our numeration, which makes the value of a figure, placed on the right of another, ten times smaller, *tenths*

naturally take their place on the right of units, then *hundredths* on the right of tenths, and so on; but, that the figures expressing decimal parts may not be confounded with those expressing whole units, a comma† is placed on the right of units. To express, for instance, 34 units and 27 hundredths, we write 34,27. If there be no units, their place is supplied by a cipher, and the same is done for all the decimal parts, which may be wanting between those enunciated in the given number.

Thus 19 hundredths are written 0,19,
304 thousandths 0,304,
3 thousandths 0,003.

85. If the expressions for the above decimal fractions be compared with the following, $\frac{19}{100}$, $\frac{304}{1000}$, $\frac{3}{1000}$, drawn from the general manner of representing a fraction, it will be seen, that *to represent in an entire form a decimal fraction, written as a vulgar fraction, the numerator of the fraction must be taken as it is, and placed after the comma in such a manner, that it may have as many figures as there are ciphers after the unit in the denominator.*

Reciprocally, *to reduce a decimal fraction, given in the form of a whole number, to that of a vulgar fraction, the figures that it contains, must receive, for a denominator, an unit followed by as many ciphers, as there are figures after the comma.*

Thus the fractions, 0,56, 0,036, are changed into $\frac{56}{100}$ and $\frac{36}{1000}$.

86. *An expression, in figures, of numbers containing decimal parts, is read by enunciating, first, the figures placed on the left of the point, then those on the right, adding to the last figure of the latter the denomination of the parts, which it represents.*

The number 26,736 is read 26 and 736 thousandths;
the number 0,0673 is read 673 ten thousandths,
and 0,0000673 is read 673 ten millionths.

---

† In English books on mathematics, and in those that have been written in the United States, decimals are usually denoted by a point, thus 0·19; but the comma is on the whole in the most general use; it is accordingly adopted in this and the subsequent treatises to be published at Cambridge.

87. As decimal figures take their value entirely from their position relative to the comma, it is of no consequence whether we write or omit any number of ciphers on their right. For instance, 0,5 is the same as 0,50; and 0,784 is the same as 0,78400; for, in the first instance, the number, which expresses the decimal fraction, becomes by the addition of a 0 ten times greater, but the parts become hundredths, and consequently on this account are ten times less than before; in the second instance, the number, which expresses the fraction, becomes a hundred times greater than before, but the parts become hundred thousandths, and, consequently, are a hundred times smaller than before. This transformation, then, becomes the same as that which takes place with respect to a vulgar fraction, when each of its terms is multiplied by the same number; and if the ciphers be suppressed, it is the same as dividing them by the same number.

88. The addition of decimal fractions and numbers accompanying them, needs no other rule than that given for the whole numbers, since the decimal parts are made up one from the other, ascending from right to left, in the same manner as whole units.

For instance, let there be the numbers 0,56, 0,003, 0,958; disposing them as follows,

	0,56
	0,003
	0,958
Sum	1,521

we find, by the rule of article 12, that their sum is 1,521.

Again. let there be the numbers 19,35, 0,3, 48,5, and 110,02, which contain also whole units, they will be disposed thus;

	19,35
	0,3
	48,5
	110,02
Sum	178,17

and their sum will be 178,17.

In general, *the addition of decimal numbers is performed like* *that of whole numbers, care being taken to place the comma in the sum, directly under the commas in the numbers to be added.*

Examples for practice.

Add 4,003, 54,9, 3,21, 6,7203. *Ans.* 68,8333.
Add 409,903, 107,7842, 6,1043, 10,2974. *Ans.* 534,0889.
Add 427, 603,04, 210,15, 3,364, ,021. *Ans.* 1243,575.

89. The rules prescribed for the subtraction of whole numbers apply also, as will be seen, to decimals. For instance, let 0,3697 be taken from 0,62; it must first be observed, that the second number, which contains only hundredths, while the other contains ten thousandths, can be converted into ten thousandths by placing two ciphers on its right (87), which changes it into 0,6200.

The operation will then be arranged thus;

	0,6200
	0,3697
Difference	0,2503

and, according to the rule of article 17, the difference will be 0,2503.

Again, let 7,364 be taken from 9,1457; the operation being disposed thus;

	9,1457
	7,3640
Difference	1,7817

the above difference is found. It would have been just as well if no cipher had been placed at the end of the number to be subtracted, provided its different figures had been placed under the corresponding orders of units or parts, in the upper line.

In general, *the subtraction of decimal numbers is performed like that of whole numbers, provided that the number of decimal figures, in the two given numbers, be made alike, by writing on the right of that, which has the least, as many ciphers as are necessary; and that the comma in the difference is put directly under those of the given numbers.*

Examples for practice.

From 304,567 take 158,632.	*Ans.* 145,935.
From 215,003 take 1,1034.	*Ans.* 213,8996.
From 1 take ,9993.	*Ans.* 0,0007.
From 68,8333 take ,00042.	*Ans.* 68,83288.

The methods of proving addition and subtraction of decimals are the same as those for the addition and subtraction of whole numbers.

90. As the comma separates the collections of entire units from the decimal parts, by altering its place, we necessarily change the value of the whole. By moving it towards the right, figures, which are contained in the fractional part, are made to pass into that of whole numbers, and consequently the value of the given number is increased. On the contrary, by moving the comma towards the left, figures, which were contained in the part of whole numbers, are made to pass into that of fractions, and consequently the value of the given number is diminished.

The first change makes the given number, ten, a hundred, a thousand, &c. times greater than before, according as the comma is removed one, two, three, &c. placed towards the right, because for each place that the comma is thus removed, all the figures advance with respect to this comma one place towards the left, and consequently assume a value ten times greater than they had before.

If, for example, in the number 134,28, the point be placed between the 2 and the 8, we shall have 1342,8, the hundreds will have become thousands, the tens hundreds, the units tens, the tenths units, and the hundredths tenths. Every part of the number having thus become ten times greater, the result is the same as if it had been multiplied by ten.

The second change makes the given number ten, a hundred, a thousand, &c. times smaller than it was before, according as the comma is removed one, two, three, &c. places towards the left, because for each place that the comma is thus removed, all the figures recede, with respect to this comma. one place further to the right, and consequently have a value ten times less than they had before.

If, in the number 134,28, the point be placed between the 3 and 4, we shall have 13,428; the hundreds will become tens, the tens units, the units tenths, the tenths hundredths, and the hundredths thousandths; every part of the number having thus become ten times smaller, the result is the same as if a tenth part of it had been taken, or as if it had been divided by ten.

91. From what has been said, it will be easy to perceive the advantage, which decimal fractions have over vulgar fractions; all the multiplications and divisions, which are performed by the denominator of the latter, are performed with respect to the former, by the addition or suppression of a number of ciphers, or by simply changing the place of the comma. By adapting these modifications to the theory of vulgar fractions, we thence immediately deduce that of decimals, and the manner of performing the multiplication and division of them; but we can also arrive at this theory directly by the following considerations.

Let us first suppose only the multiplicand to have decimal figures. If the comma be taken away, it will become ten, a hundred, a thousand, &c. times greater, according to the number of decimal figures; and in this case the product given by multiplication will be a like number of times greater than the one required; the latter will then be obtained by dividing the former by ten, a hundred, a thousand, &c. which may be done by separating on the right(90) as many decimal figures, as there are in the multiplicand.

If, for instance, 34,137 were to be multiplied by 9, we must first find the product of 34137 by 9, which will be 307233; and, since taking away the comma renders the multiplicand a thousand times greater, we must divide this product by a thousand, or separate by a comma its three last figures on the right; we shall thus have 307,233.

In general, *to multiply, by a whole number, a number accompanied by decimals, the comma must be taken away from the multiplicand, and as many figures separated for decimals, on the right of the product, as are contained in the multiplicand.*

Examples for practice.

Multiply 231,415 by 8. *Ans.* 1851,320.
Multiply 32,1509 by 15. *Ans.* 482,2635.
Multiply 0,840 by 840. *Ans.* 705,600.
Multiply 1,236 by 13. *Ans.* 16,068.

92. When the multiplier contains decimal figures, by suppressing the comma, it is made ten, a hundred, a thousand, &c. times greater according to the number of decimal figures. If used in this state, it will evidently give a product, ten, a hundred, a thousand, &c. times greater than that which is required, and consequently the true product will be obtained by dividing by one of these numbers, that is, by separating, on the right of it, as many decimal figures as there are in the multiplier, or by removing the comma a like number of places towards the left(90), in case it previously existed in the product on account of decimals in the multiplicand. For instance, let 172,84 be multiplied by 36,003; taking away the comma in the multiplier only, we shall have, according to the preceding article, the product 6222758,52; but, the multiplier being rendered a thousand times too great, we must divide this product by a thousand, or remove the comma three places towards the left, and the required product will then be 6222,75852, in which there must necessarily be as many decimal figures as there are in both multiplicand and multiplier.

In general, *to multiply one by the other, two numbers accompanied by decimals, the comma must be taken away from both, and as many figures separated for decimals, on the right o the product, as there are in both the factors.*

In some cases it is necessary to put one or more ciphers on the left of the product, to give the number of decimal figures required by the above rule. If, for example, 0,624 be multiplied by 0,003; in forming at first the product of 624 by 3, we shall have the number 1872, containing but 4 figures, and as 6 figures must be separated for decimals, it cannot be done except by placing on the left three ciphers, one of which must occupy the place of units, which will make 0,001872.

Examples for practice.

Multiply 223,86 by 2,500. *Ans.* 559,65000.
Multiply 35,640 by 26,18. *Ans.* 933,05520.
Multiply 8,4960 by 2,618. *Ans.* 22,2425280.
Multiply 0,5236 by 0,2808. *Ans.* 0,14702688.
Multiply 0,11785 by 0,27. *Ans.* 0,0318195.

93. It is evident (36), that the quotient of two numbers does not depend on the absolute magnitude of their units, provided that this be the same in each; if then, it be required to divide 451,49 by 13, we should observe that the former amounts to 45149 hundredths, and the latter to 1300 hundredths, and that these last numbers ought to give the same quotient, as if they expressed units. We shall thus be led to suppress the point in the first number, and to put two ciphers at the end of the second, and then we shall only have to divide 45149 by 1300, the quotient of which division will be $34\frac{949}{1300}$.

Hence we conclude, that, *to divide, by a whole number, a number accompanied by decimal figures, the comma in the dividend must be taken away, and as many ciphers placed at the end of the divisor, as the dividend contains decimal figures, and no alteration in the quotient will be necessary.*

94. When both dividend and divisor are accompanied by decimal figures, we must, before taking away the comma, reduce them to decimals of the same order, by placing at the end of that number, which has the fewest decimal figures, as many ciphers as will make it terminate at the same place of decimals as the other, because then the suppression of the comma renders both the same number of times greater.

For instance, let 315,432 be divided by 23,4, this last must be changed into 23,400, and then 315432 must be divided by 23400; the quotient will be $13\frac{11232}{23400}$.

Thus, *to divide one by the other, two numbers accompanied by decimal figures, the number of decimal figures in the divisor and dividend must be made equal, by annexing to the one, that has the least, as many ciphers as are necessary; the point must then be suppressed in each, and the quotient will require no alteration.*

95. As we have recourse to decimals only to avoid the neces-

sity of employing vulgar fractions, it is natural to make use of decimals for approximating quotients that cannot be obtained exactly, which is done by converting the remainder into tenths, hundredths, thousandth, &c. so that it may contain the divisor; as may been in the following example;

	45149	1300
	3900	34,73
	6149	
	5200	
Remainder	949	
tenths	9490	
	9100	
hundredths	3900	
	3900	
	0	

When we come to the remainder 949, we annex a cipher in order to multiply it by ten, or to convert it into tenths; thus forming a new partial dividend, which contains 9490 tenths and gives for a quotient 7 tenths, which we put on the right of the units, after a comma. There still remains 390 tenths, which we reduce to hundredths by the addition of another cipher, and form a second dividend, which contains 3900 hundredths, and gives a quotient, 3 hundredths, which we place after the tenths. Here the operation terminates, and we have for the exact result 34,73 hundredths. If a third remainder had been left, we might have continued the operation, by converting this remainder into thousandths, and so on, in the same manner, until we came to an exact quotient, or to a remainder composed of parts so small, that we might have considered them of no importance.

It is evident, that we must always put a comma, as in the above example, after the whole units in the quotient, to distinguish them from the decimal figures, the number of which must be equal to that of the ciphers successively written after the remainders*.

---

* The problem above performed with respect to decimals, is only

Examples for practice.

Divide 6345,925	by 54,23.	*Ans.* 117,018 &c.
Divide 5673,21	by 23,0.	*Ans.* 246,660 &c.
Divide 84329907	by 627,1.	*Ans.* 134476,01 &c.
Divide 27845,96	by 9,8732.	*Ans.* 2820,3581 &c.
Divide 200,5	by 231.	*Ans.* 0,0867 &c.
Divide 10,0	by 563,0.	*Ans.* 0,00177 &c.
Divide 513,2	by 0.057.	*Ans.* 9003,50 &c.
Divide 7,25406	by 957.	*Ans.* 0,00758
Divide 0,00078759	by 0,525.	*Ans.* 0,00150 &c.
Divide 14	by 365.	*Ans.* 0,038356 &c.

96. The numerator of a fraction, being converted into decimal parts, can be divided by the denominator as in the preceding examples, and by this means the fraction will be converted into decimals. Let the fraction, for example, be $\frac{1}{8}$, the operation is performed thus;

$$\begin{array}{r|l} 1 & 8 \\ \hline 10 & 0,125 \\ 8 & \\ \hline 20 & \\ 16 & \\ \hline 40 & \\ 40 & \\ \hline 0 & \end{array}$$

Again, let the fraction be $\frac{4}{797}$; the numerator must be converted into thousandths before the division can begin.

---

a particular case of the following more general one; *To find the value of the quotient of a division, in fractions of a given denomination;* to do this we convert the dividend into a fraction of the same denomination by multiplying it by the given denominator. Thus, in order to find in fifteenths the value of the quotient of 7 by 3, we should multiply 7 by 15, and divide the product, 105, by 3, which gives thirty-five fifteenths, or $\frac{35}{15}$ for the quotient required.

$$
\begin{array}{l|l}
4 & 797 \quad * \\
\hline
4000 & 0{,}005018\ \&c. \\
3985 & \\
\hline
\quad 1500 & \\
\quad\ \ 797 & \\
\hline
\quad\ \ 7030 & \\
\quad\ \ 6376 & \\
\hline
\quad\quad 654 &
\end{array}
$$

Examples for practice.

Reduce $\frac{3}{4}$ to a decimal fraction. *Ans.* 0,75

Reduce $\frac{1}{2}$ to a decimal fraction. *Ans.* 0,5.

Reduce $\frac{5}{70}$ to a decimal fraction. *Ans.* 0,0714285 &c.

Reduce $\frac{5}{100}$ to a decimal fraction. *Ans.* 0,05.

Reduce $\frac{3}{9}$ to a decimal fraction. *Ans.* 0,333 &c.

97. However far we may continue the second division, exhibited above, we shall never obtain an exact quotient, because the fraction $\frac{4}{797}$ cannot, like $\frac{1}{8}$, be exactly expressed by decimals.

The difference in the two cases arises from this, that the denominator of a fraction, which does not divide its numerator, cannot give an exact quotient, except it will divide one of the numbers 10, 100, 1000, &c. by which its numerator is successively multiplied, because it is a principle, which will be found demonstrated in Algebra, that no number will divide a product except its factors will divide those of the product; now the numbers 10, 100, 1000, &c. being all formed from 10, the factors of which are 2 and 5, they cannot be divided except by

* It may also be proposed to convert a given fraction into a fraction of another denomination, but smaller than the first, for instance, $\frac{3}{4}$ into seventeenths, which will be done by multiplying 3 by 17 and dividing the product by 4. In this manner we find $\frac{51}{4}$ seventeenths, or $\frac{12}{17}$ and $\frac{3}{4}$ of a seventeenth; but $\frac{3}{4}$ of $\frac{1}{17}$ is equivalent to $\frac{3}{68}$. The result then, $\frac{12}{17}$, is equal to $\frac{3}{4}$, wanting $\frac{3}{68}$.

This operation and that of the preceding note depend on the same principle, as the corresponding operation for decimal fractions.

numbers formed from these same factors; 8 is among these, being the product of 2 by 2 by 2.

Fractions, the value of which cannot be exactly found by decimals, present in their approximate expression, when it has been carried sufficiently far, a character which serves to denote them; this is the periodical return of the same figures.

If we convert the fraction $\frac{12}{37}$ into decimals, we shall find it 0,324324 . . . . ., and the figures 3, 2, 4, will always return in the same order, without the operation ever coming to an end.

Indeed, as there can be no remainder in these successive divisions except one of the series of whole numbers, 1, 2, 3, &c. up to the divisor, it necessarily happens, that, when the number of divisions exceeds that of this series, we must fall again upon some one of the preceding remainders, and consequently the partial dividends will return in the same order. In the above example three divisions are sufficient to cause the return of the same figures; but six are necessary for the fraction $\frac{1}{7}$, because in this case we find, for remainders, the six numbers which are below 7, and the result is 0,1428571 . . . The fraction $\frac{1}{3}$ leads only to 0,3333 . . . . .

98. The fractions, which have for a denominator any number of 9s, have no significant figure in their periods except 1;

$\frac{1}{9}$ gives 0,11111 . . .
$\frac{1}{99}$ 0,010101 . . . . .
$\frac{1}{999}$ 0,001001001 . . . . .

and so with the others, because each partial division of the numbers 10, 100, 1000, &c. always leaves unity for the remainder.

Availing ourselves of this remark, we pass easily from a periodical decimal, to the vulgar fraction from which it is derived. We see, for example, that 0,33333 . . . . . amounts to the same as 0,11111 . . . . . multiplied by 3, and as this last decimal is the development of $\frac{1}{9}$, or $\frac{1}{9}$ reduced to a decimal, we conclude, that the former is the development of $\frac{1}{9}$ multiplied by 3, or $\frac{3}{9}$, or lastly, $\frac{1}{3}$.

When the period of the fraction under consideration consists of two figures, we compare it with the development of $\frac{1}{99}$, and with that of $\frac{1}{999}$, when the period contains three figures, and so on.

If we had, for example, 0,324324, it is plain that this fraction may be formed by multiplying 0,001001 . . . . . by 324; if we multiply then $\frac{1}{999}$, of which 0,001001 is the development, by 324, we obtain $\frac{324}{999}$, and dividing each term of this result by 27, we come back again to the fraction $\frac{12}{37}$.

In general, *the vulgar fraction from which a decimal fraction arises, is formed by writing, as a denominator, under the number, which expresses one period, as many 9s, as there are figures in the period.*

If the period of the fraction does not commence with the first decimal figure, we can for a moment change the place of the point, and put it immediately before the first figure of the period and beginning with this figure, find the value of the fraction, as if those figures on the left were units; nothing then will be necessary except to divide the result by 10, 100, 1000, &c. according to the number of places the point was moved towards the right.

For instance, the fraction 0,324141, is first to be written 32,4141; the part 0,4141 being equivalent to $\frac{41}{99}$, we shall have $32\frac{41}{99}$, which is to be divided by 100, because the point was moved two places towards the left; it will consequently become $\frac{32}{100}$ and $\frac{41}{9900}$, or by reducing the two parts to the same denominator, and adding them, $\frac{3209}{9900}$, a fraction which will reproduce the given expression.

*Examples for practice.**

Reduce $0,\dot{1}\dot{8}$ to the form of a vulgar fraction. *Ans.* $\frac{2}{11}$.

Reduce $0,\dot{7}\dot{2}$ to the form of a vulgar fraction. *Ans.* $\frac{8}{11}$.

Reduce $0,8\dot{3}$ to the form of a vulgar fraction. *Ans.* $\frac{5}{6}$.

Reduce $0,24\dot{1}\dot{8}$ to the form of a vulgar fraction. *Ans.* $\frac{1208}{4995}$.

Reduce $0,27\dot{5}46\dot{3}$ to the form of a vulgar fraction. *Ans.* $\frac{22953}{83325}$.

Reduce $0,91\dot{6}$ to the form of a vulgar fraction. *Ans.* $\frac{11}{12}$.

---

* In these examples, the better to distinguish the period, a point is placed over it, if it be a single figure, and over the first and last figure, if it consist of more than one.

To form a correct idea of the nature of these fractions it is sufficient to consider the fraction 0,999. In trying to discover its original value we find that it answers to 9 divided by 9, that is, to unity; nevertheless, at whatever number of figures we stop in its expression, it will never make an unit. If we stop at the first figure, it wants $\frac{1}{10}$ of an unit; if at the second, it wants $\frac{1}{100}$; if at the third, it wants $\frac{1}{1000}$, and so on; so that we can arrive as near to unity as we please, but can never reach it. Unity then in this case is nothing but a *limit*, to which 0,999 . . . . . . continually approaches the nearer the more figures it has.

99. The preceding part of this work contains all the rules absolutely essential to the arithmetic of abstract numbers, but to apply them to the uses of society it is necessary to know the different kinds of units, which are used to compare together, or ascertain the value of quantities, under whatever form they may present themselves. These units, which are the measures in use, have varied with times and places, and their connexion has been formed only by degrees, accordingly as necessity and the progress of the arts and sciences have required greater exactness in the valuation of substances, and the construction of instruments.

TABLES OF COIN, WEIGHT, AND MEASURE.

Denominations of Federal money, as determined by an act of Congress, Aug. 8, 1786†.

			Marked.
10 mills	make	one cent	c.
10 cents		one dime	d.
10 dimes		one dollar	$.
10 dollars		one eagle	E.

† The coins of federal money are two of gold, four of silver, and two of copper. The gold coins are an *eagle* and *half-eagle*; the silver, a *dollar*, *half-dollar*, *double dime*, and *dime*; and the copper a *cent* and *half-cent*. The standard for gold and silver is eleven parts fine and one part alloy. The weight of fine gold in the eagle is 246,268 grains; of fine silver in the dollar, 375,64 grains; of copper

English Money.

4 farthings make	1 penny	£ denotes	pounds.
12 pence	1 shilling	*s*	shillings.
20 shillings	1 pound	*d*	pence.
		q	quarters or farthings.

TROY WEIGHT.

24 grains make	1 penny-weight, marked	grs. dwt.
20 dwt.	1 ounce,	oz.
12 oz.	1 pound,	℔.

By this weight are weighed jewels, gold, silver, corn, bread, and liquors.

APOTHECARIES' WEIGHT.

20 grains make	1 scruple, marked	gr. sc.
3 sc.	1 dram,	dr. or ʒ.
8 dr.	1 ounce,	oz. or ℥.
12 oz.	1 pound,	℔.

Apothecaries use this weight in compounding their medicines;

---

in 100 cents, $2\frac{1}{4}$ lb. avoirdupois. The fine gold in the half-eagle is half the weight of that in the eagle; the fine silver in the half-dollar, half the weight of that in the dollar, &c. The denominations less than a dollar are expressive of their values; thus, *mill* is an abbreviation of *mille*, a thousand, for 1000 mills are equal to 1 dollar; *cent*, of *centum*, a hundred, for 100 cents are equal to 1 dollar; a *dime* is the French of *tithe*, the tenth part, for 10 dimes are equal to 1 dollar.

The mint price of uncoined gold, 11 parts being fine and 1 part alloy, is 209 dollars, 7 dimes, and 7 cents per lb. Troy weight; and the mint price of uncoined silver, 11 parts being fine and 1 part alloy, is 9 dollars, 9 dimes, and 2 cents per lb. Troy.

In practical treatises on arithmetic, may be found rules for reducing the Federal Coin, the currencies of the several United States, and those of foreign countries, each to the *par* of all the others. It may be sufficient here to observe respecting the currencies of the several states, that a dollar is considered as 6s. in New-England and Virginia; 8s. in New-York and North Carolina; 7s. 6d. in New-Jersey, Pennsylvania, Delaware, and Maryland; and 4s. 8d. in South Carolina and Georgia; the denomination of shilling varying its value accordingly.

but they buy and sell their drugs by Avoirdupois weight. Apothecaries' is the same as Troy weight, having only some different divisions.

AVOIRDUPOIS WEIGHT.

16	drams make	1 ounce, marked	dr. oz.
16	ounces	1 pound,	℔.
28	℔.	1 quarter,	qr.
4	quarters	1 hundred weight,	cwt.
20	cwt.	1 ton,	T.

By this weight are weighed all things of a coarse or drossy nature ; such a butter, cheese, flesh, grocery wares, and all metals, except gold and silver.

DRY MEASURE.

36 bushels [illegible]

		Marked			Marked
2 pints make	1 quart,	pts. qts.	8 bushels	1 quarter,	qr.
2 quarts	1 pottle,	pot.	5 quarters	1 wey *or* load,	wey.
2 pottles	1 gallon,	gal.	4 bushels	1 coom *or* carnock,	co.
2 gallons	1 peck,	pe.	2 cooms	a seam *or* quarter.	
4 pecks	1 bushel,	bu.	6 seams	1 wey.	
2 bushels	1 strike,	str.	$1\frac{2}{3}$ weys	1 last,	L.

The diameter of a Winchester bushel is $18\frac{1}{2}$ inches, and its depth 8 inches.—And one gallon by dry measure contains $268\frac{4}{5}$ cubic inches.

By this measure, salt, lead, ore, oysters, corn, and other dry goods are measured.

ALE AND BEER MEASURE.

		Marked			Marked
2 pints make	1 quart,	pts. qts.	2 firkins	1 kilderkin,	kil.
4 quarts	1 gallon,	gal.	2 kilderkins	1 barrel,	bar.
8 gallons	1 firkin of Ale,	fir.	3 kilderkins	1 hogshead,	hhd.
9 gallons	1 firkin of Beer,	fir.	3 barrels	1 butt,	butt.

The ale gallon contains 282 cubic inches. In London the ale firkin contains 8 gallons, and the beer firkin 9 ; other measures being in the same proportion.

WINE MEASURE.

			Marked
2 pints	make	1 quart,	pts. qts.
4 quarts		1 gallon,	gal.
42 gallons		1 tierce,	tier.
63 gallons		1 hogshead,	hhd.
84 gallons		1 puncheon,	pun.
2 hogsheads		1 pipe *or* butt,	p. *or* b.
2 pipes		1 tun,	T.
18 gallons		1 runlet,	run.
$31\frac{1}{2}$ gallons		1 barrel,	bar.

By this measure, brandy, spirits, perry, cider, mead, vinegar, and oil are measured.

231 cubic inches make a gallon, and 10 gallons make an anchor.

CLOTH MEASURE.

			Marked
$2\frac{1}{4}$ inches	make	1 nail,	nls.
4 nails		1 quarter,	qrs.
4 quarters		1 yard,	yds.
3 qrs.		1 ell Flemish,	Ell Fl.
5 qrs.		1 ell English,	Ell Eng.
6 qrs.		1 ell French,	Ell Fr.

LONG MEASURE.

			Marked
3 barley corns	make	1 inch,	bar. c. in.
12 inches		1 foot,	ft.
3 feet		1 yard,	yd.
6 feet		1 fathom,	fath
$5\frac{1}{2}$ yards		1 pole,	pol.
40 poles		1 furlong,	fur.
8 furlongs		1 mile,	mls.
3 miles		1 league,	l.
60 geographical miles, *or* $69\frac{1}{5}$ statute miles		1 degree nearly,	deg. *or* °
360 degrees		the circumference of the earth.	

Also, 4 inches make 1 hand.

5 feet	1 geometrical space.	
6 points	1 line.	
12 lines	1 inch.	

TIME.

			Marked
60 seconds	make	1 minute,	s. *or* ″ m. *or* ′
60 minutes		1 hour,	h. *or* °
24 hours		1 day,	d.
7 days		1 week,	w.
4 weeks		1 month,	m.
13 months, 1 day, and 6 hours, *or* 365 days and 6 hours,		1 Julian year,	Y.

100. It is evident, that if the several denominations of money, weight and measure proceeded in a decimal ratio, the fundamental operations might be performed upon these, as upon abstract numbers. This may be shown by a few examples in Federal Money. If it were required to find the sum of \$46,85

and \$256,371, we should place the numbers of the same denomination in the same column, and add them together as in whole numbers; thus,

4685
256371
303221

and the answer may be read off in either or all the denominations; we may say 30 eagles 3 dollars 22 cents 1 mill, or 303 dollars 221 thousandths, or 30322 cents and 1 tenth, or 303221 mills. It is usual to consider the dollars as whole numbers, and the following denominations as decimals. The operation then becomes the same as for decimals.

Examples.

Add	\$34,123	Add	\$456,78
	1,178		49,83
	78,001		0,22
	61,789		7854,394
Sum	\$175,091	Sum	\$8361,224

From	\$542,76	From	\$527,839
Subtract	239,481	Subtract	22,94
Rem.	303,279	Rem.	504,899

Multiply	\$6,347	by \$4,532.	*Ans.* \$28,764604.
Divide	\$28,764604	by \$4,532.	*Ans.* \$6,347.
Divide	\$20	by \$2000.	*Ans.* \$0,01.

REDUCTION.

101. WHEN the different denominations do not proceed in a decimal ratio, they may all be *reduced* to one denomination, and then the fundamental operations may be performed upon this, as upon an abstract number. If, for example, the sum to be operated upon were £4 15s. 9d. this may easily be expressed in

pence. As 1 pound is 20 shillings, 4 pounds will be 4 times 20, or 80 shillings. If to this we add the 15s. we shall have 95s. 9d. equivalent to the above. But as 1 shilling is equal to 12 pence, 95s. will be equal to 95 times 12 or 1140 pence. Adding 9 to this, we shall have 1149 pence as an equivalent expression for £4 15s. 9d. We may now make use of this number as if it had no relation to money or any thing else; and the result obtained may be converted again into the different denominations by reversing the process above pursued. If it were proposed to multiply this sum by another number, 37 for instance, we should find the product of these two numbers in the usual way; thus,

$$\begin{array}{r} 1149 \\ 37 \\ \hline 8043 \\ 3447 \\ \hline 42513 \end{array}$$

42513 is, therefore, equal to 37 times £4 15s. 9d. expressed in pence; to find the number of pounds and shillings contained in this, we first obtain the number of shillings by dividing it by 12, which gives 3542, and then the number of pounds by dividing this last by 20; thus,

$$\begin{array}{r|l} 42513 & 12 \\ \cline{2-2} 65 & 3542 \\ 51 & \\ 33 & \\ 9 & \\ \hline \end{array} \qquad\qquad \begin{array}{r|l} 354,2 & 20 \\ \cline{2-2} 15 & 177 \\ 14 & \\ 2 & \end{array}$$

42513 pence then is equal to 3542 shillings and 9 pence, or to 177 pounds 2 shillings and 9 pence. Whence 37 times £4 15s. 9d. is equal to £177 2s. 9d.

It may be remarked, that *shillings are converted into pounds by separating the right hand figure and dividing those on the left by* 2, prefixing the remainder, if there be one, to the figure separated for the entire shillings, that remain. This amounts to dividing, first, by 10 (90), and then that quotient by 2. If 10 shillings made a pound, dividing by 10 would give the number of pounds, but as 10 shillings are only half a pound, half this number will be the number of pounds.

By a method similar to that above given, we reduce other denominations of money and the different denominations of the several weights and measures to the lowest respectively. If it were required to find how many grains there are in 2lb. 4oz. 17dwt. 5grs. Troy, we should proceed thus,

```
         lb.   oz.   dwt.   grs.
          2     4     17     5
         12
        ----
         24
          4
        ----
         28
         20
        ----
        560
         17
        ----
        577
         24
        ----
       2308
      1154
      -----
      13848
          5
      -----
Ans.  13853
```

By dividing 13853 by 24, and the quotient thence arising by 20, and this second quotient by 12, we shall evidently obtain the number of pounds, ounces, pennyweights and grains in 13853 grains. The operation may be seen below.

```
13853 | 24
120   |----
----    577 | 20
 185    40  |----
 168    ---    28 | 12
 ---    177    24 |----
  173   160    ---    2
  168   ---     4
  ---    17
    5
                 lb.  oz.  dwt.  gr.
         Result   2    4    17    5
```

These examples will be sufficient to establish the following general rules, namely;

To reduce a compound number to the lowest denomination contained in it, multiply the highest by so many as one of this denomination makes of the next lower, and to the product add the number belonging to the next lower; proceed with each succeeding denomination in a similar manner, and the last sum will be the number required.

To reduce a number from a lower denomination to a higher, divide by so many as it takes of this lower denomination to make one of the higher, and the quotient will be the number of the higher; which may be further reduced in the same manner if there are still higher denominations, and the last quotient together with the several remainders will be equivalent to the number to be reduced.

Examples for practice.

In 59lb. 13dwt. 5gr. how many grains? *Ans.* 340157.

In 8012131 grains how many pounds, &c.?
Ans. 1390℔. 11oz. 18dwt. 19gr.

In 121*l.* 0s. $9\frac{1}{2}$d. how many half pence? *Ans.* 58099.

In 58099 half pence how many pounds &c.? *Ans.* 121*l.* 0s. $9\frac{1}{2}$d.

In 48 guineas at 28s. each how many $4\frac{1}{2}$ pence?
Ans. 3584.

In one year of 365d. 5h. 48′ 48″ how many seconds?
Ans. 31556928.

102. When we have occasion to make use of a number consisting of several denominations as an abstract number, instead of reducing the several parts to the lowest denomination contained in it, we may reduce all the lower denominations to a fraction of the highest. Taking the sum before used, namely, 4l. 15s. 9d. we reduce the lower denominations to the higher, as in the last article by division. The number of pence 9, or $\frac{9}{1}$, is divided by 12, by multiplying the denominator by this number (54), we have thus, $\frac{9}{12}$s. which being added to 15s. or $\frac{180}{12}$s. the whole number being reduced to the form of a fraction of the same denominator, we have $\frac{180}{12}$ and $\frac{9}{12}$, which being added, make $\frac{189}{12}$. This is further reduced to pounds by dividing it by 20,

that is, by multiplying the denominator by 20 (54), which gives $\frac{189}{240}$. Whence £4 15s. 9d. is equal to £$4\frac{189}{240}$, or £$\frac{1149}{240}$. This may now be used like any other fraction, and the value of the result found in the different denominations. If we multiply it by 37, we shall have £$\frac{42513}{240}$, or £$177\frac{33}{240}$; and £ $\frac{33}{240}$, reduced to shillings by multiplying the numerator by 20, or dividing the denominator by this number, gives $\frac{33}{12}$s. or $2\frac{9}{12}$s. or 2s. 9d.

From the above example we may deduce the following general rules, namely,

To reduce the several parts of a compound number to a fraction of the highest denomination contained in it, make the lowest term the numerator of a fraction, having for its denominator the number which it takes of this denomination to make one of the next higher, and add to this the next term reduced to a fraction of the same denomination, then multiply the denominator of this sum by so many as make one of the next denomination, and so on through all the terms, and the last sum will be the fraction required†.

To find the value of a fraction of a higher denomination in terms of a lower, multiply the numerator of the fraction by so many as make one of the lower denomination, and divide the product by the denominator, and the quotient will be the entire number of this denomination, the fractional part of which may be still further reduced in the same manner.

To reduce 2w. 1d. 6h. to the fraction of a month.

6h. is $\frac{6}{24}$ of a day, and being added to one day, or $\frac{24}{24}$d. gives $\frac{30}{24}$d. the denominator of which being multiplied by 7, it becomes $\frac{30}{168}$w. and being added to 2 weeks or twice $\frac{168}{168}$w. gives $\frac{366}{168}$w. If we now multiply the denominator of this by 4, we shall have $\frac{366}{672}$ of a month, as an equivalent expression for 2w. 1d. 6h.

To find the value of $\frac{5}{7}$ of a mile in furlongs, poles, &c.

---

† It will often be found more convenient to reduce the several parts of the compound number to the lowest denomination, as by the preceding article for a numerator, and to take for the denominator so many of this denomination as it takes to make one of that, to which the expression is to be reduced; thus 4*l.* 15s. 9d. being 1149d. is equal to $\frac{1149}{240}$*l.* because 1d. is $\frac{1}{240}$*l.*

```
   5
   8
  ——
  40 | 7
  35 |——
  —— | 5
   5
  40
  ——   7
 200  ——
  14  28
 ———
  60
  56
  ——
   4
  5½    7
  ——   ——
  22 | 3⅐
  21 |
  ——
   1
```

Ans. 5fur. 28pls. $3\frac{1}{7}$yds.

Reduce 13s. 6d. 2q. to the fraction of a pound.

Ans. £$\frac{650}{960}$, or £$\frac{65}{96}$.

Reduce 6fur. 26pls. 3yds. 2ft. to the fraction of a mile.

Ans. $\frac{4400}{5280}$, or $\frac{5}{6}$.

Reduce 7oz. 4dwt. to the fraction of a pound, Troy. *Ans.* $\frac{3}{5}$.

What part of a mile is 6fur. 16pls.? *Ans.* $\frac{4}{5}$.

What part of a hogshead is 9 gallons? *Ans.* $\frac{1}{7}$.

What part of a day is $\frac{3}{13}$ of a month? *Ans.* $\frac{84}{13}$.

What part of a penny is $\frac{1}{18}$ of a pound? *Ans.* $\frac{40}{3}$.

What part of a cwt. is $\frac{6}{7}$ of a pound, Avoirdupois? *Ans.* $\frac{3}{392}$.

What part of a pound is $\frac{2}{3}$ of a farthing? *Ans.* $\frac{1}{1440}$.

What is the value of $\frac{3}{5}$ of a pound, Troy? *Ans.* 7oz. 4dwt.

What is the value of $\frac{4}{7}$ of a pound, Avoirdupois?

Ans. 9oz. $2\frac{2}{7}$dr.

What is the value of $\frac{7}{9}$ of a cwt.? *Ans.* 3qrs. 3lb. 1oz. $12\frac{4}{9}$dr.

What is the value of $\frac{3}{17}$ of a mile?

Ans. 1fur. 16pls. 2yds. 1ft. $9\frac{3}{17}$ in.

What is the value of $\frac{7}{13}$ of day? *Ans.* 12h. 55′ $23\frac{1}{13}$″.

The several parts of a compound number may also be reduced to the form of a decimal fraction of the highest denomination contained in it, by first finding the value of the expression in a vulgar fraction, as in the last article, and then reducing this to a decimal, or more conveniently by changing the terms to be reduced into decimals parts, and dividing the numerator instead of multiplying the denominator by the numbers successively employed in raising them to the required denomination.

If we take the sum already used, namely, £4 15s. 9d. the pence, 9, may be written $\frac{90}{10}$, or $\frac{900}{100}$. the numerator of which admits of being divided by 12 without a remainder. It is thus reduced to shillings and becomes $\frac{75}{100}$s. or 0,75s. which added to the 15s. makes 15,75s. or reducing the 15 to the same denomination, $\frac{1575}{100}$, or $\frac{157500}{10000}$; and this is reduced to pounds, by dividing it by 20, the result of which is $\frac{7875}{10000}$, or 0,7875. 4*l.* 15s. 9d. therefore may be expressed in one denomination, thus, 4,7875*l.* and in this state it may be used like any other number consisting of an entire and fractional part. If it be multiplied by 37, we shall have for the product 177,1375*l.* This decimal of a pound may be reduced to shillings and pence, by reversing the above process, or by multiplying successively by 20 and then by 12.

$$\begin{array}{r} 0,1375 \\ 20 \\ \hline 2,7500 \\ 12 \\ \hline 9,0000 \end{array}$$

The product therefore of 4*l.* 15s. 9d. by 37 is 177*l.* 2s. 9d. as before obtained.

The operation, just explained, admits of a more convenient disposition, as in the following example.

To reduce 19s. 3d. 3q. to the decimal of a pound.

$$\begin{array}{r|l} 4 & 3,00 \\ 12 & 3,75000 \\ 20 & 19,312500 \\ \hline & 0,965625 \end{array}$$

Proceeding as before, we reduce the farthings, 3, considered as $\frac{300}{100}$q. to hundredths of a penny by dividing by the figure on the left, 4, and place the quotient, 75, as a decimal on the right of the pence; we then take this sum, considered as $\frac{375}{100}$d. or $\frac{3750}{1000}$d. that is, annexing as many ciphers as may be necessary, and divide it by 12, which brings it into decimals of a shilling. Lastly, the shillings and parts of a shilling, 19,3125s. considered as $\frac{19312500}{1000000}$ s. are reduced to decimals of a pound by dividing by 20, which gives the result above found.

We may proceed in a similar manner with other denominations of money and with those of the several weights and measures. One example in these will suffice as an illustration of the method.

To reduce 17pls. 1ft. 6in. to the decimal of a mile.

$$\begin{array}{r|l} 12 & 6 \\ 16{,}5 & 1{,}5 \\ 320 & 17{,}\dot{0}\dot{9} \\ \hline & 0{,}00531531 \text{ \&c.} \end{array}$$

The decimal in this, as in many other cases, becomes periodical (97).

From what has been said, the following rules are sufficiently evident. *To reduce a number from a lower denomination to the decimal of a higher, we first change it, or suppose it to be changed into a fraction, having* 10, *or some multiple of* 10, *for its denominator, and divide the numerator by so many as make one of this higher denomination, and the quotient is the required decimal; which, together with the whole number of this denomination, may again be converted into a fraction, having* 10 *or a multiple of* 10 *for its denominator, and thus by division be reduced to a still higher name, and so on.*

Also, *to reduce a decimal of a higher denomination to a lower, we multiply it by so many as one makes of this lower, and those figures which remain on the left of the comma, when the proper number is separated for decimals* (91), *will constitute the whole number of this denomination, the decimal part of which may be still further reduced, if there be lower denominations, by multiplying it by the number which one makes of the next denomination, and so on.*

It may be proper to add in this place, that shillings, pence and farthings may readily be converted into the fraction of a pound, and the fraction of a pound reduced to shillings, pence and farthings, without having recourse to the above rules. As shillings are so many twentieths of a pound, by dividing any given number of shillings by 2, we convert them into decimals of a pound, thus, 15s. which may be written $\frac{15}{20}l.$ or $\frac{150}{200}l.$ being divided by 2 give 75 hundredths, or 0,75 of a pound. Also, as farthings are so many 960ths of a pound, one pound being equal to 960 farthings, the pence converted into farthings and united with those of this denomination, may be written as so many 960ths of a pound. If now we increase the numerator and denominator one twenty fourth part, we shall convert the denominator into thousandths, and the numerator will become a decimal.

Whence, *to convert shillings, pence and farthings, into the decimal of a pound, divide the shillings by* 2, *adding a cipher when necessary, and let the quotient occupy the first place, or first and second, if there be two figures, and let the farthings, contained in the pence and farthings, be considered as so many thousandths, increasing the number by one, when the number is nearer* 24 *than* 0, *and by* 2, *when it is nearer* 48 *than* 24, *and so on.*

Thus, to reduce 15s. 9d. to the decimal of a pound, we have,

$$\begin{array}{r} 0{,}75 \\ 37 \\ \hline 0{,}787 \end{array}$$

This result, it will be remarked, is not exactly the same as that obtained by the other method; the reason is, that we have increased the number of farthings, 36, by only one, whereas, allowing one for every 24, we ought to have increased it one and a half. Adding, therefore, a half, or 5 units of the next lower order, we shall have 0,7875, as before.

On the other hand, *the decimal of a pound is converted into the lower denominations, or its value is found in shillings, pence and farthings, by doubling the first figure for shillings, increasing it by one, when the second figure is* 5, *or more than* 5, *and considering what remains in the second and third places, as farthings, after having diminished them one for every* 24.

In addition to the rules that have been given, it may be observed, that in those cases, where it is required to reduce a number from one denomination to another, when the two denominations are not commensurable or when one will not exactly divide the other, it will be found most convenient, as a general rule, to reduce the one, or both, when it is necessary, to parts so small, that a certain number of the one will exactly make a unit of the other. If it were required, for instance, to reduce pounds to dollars, as a pound does not contain an exact number of dollars without a fraction, we first convert the pounds into shillings, and then, as a certain number of shillings make a dollar, by dividing the shillings by this number, we shall find the number of dollars required. A similar method may be pursued in other cases of a like nature, as may be seen in the following examples.

In 178 guineas at 28s. each, how many crowns at 6s. 8d.?

6s. 8d.	178	5980,8	80
12	28	48	747
80d.	1424		
	356		
	4984		
	12		
	59808		

Ans. 747 crowns and 4 shillings†.

In this case, I reduce both the guineas and the crown to pence, and then divide the former result by the latter. In dividing by 80, I first separate one figure on the right of the dividend for a decimal, which is the same as dividing it by 10, and then divide the figures on the left, or the quotient, by 8 (47), joining what remains as tens to the figures separated, to form the entire remainder, which is reduced back to the original denomination.

To reduce 137 five franc pieces to pounds, shillings, &c. the franc being valued at $0,1796.

---

† Questions of this kind may often be conveniently performed by fractions; thus, 178 guineas, or 4984s. divided by 6s. 8d. or $6\frac{2}{3}$s. or reducing the whole number to the form of a fraction, $\frac{20}{3}$s. becomes $\frac{4984}{1}$ multiplied by $\frac{3}{20}$ (74), or $\frac{14952}{20}$, or $\frac{1495,2}{2}$, which is equal to $747\frac{12}{20}$; and $\frac{12}{20}$, or $\frac{3}{5}$, of 6s. 8d. is 3 times $\frac{1}{5}$ of 80d. or 48d. or 4s.

0,1795	73,8156 | 20
5	36,9078
0,8980	20
137	18,1560
6286	12
2694	1,8720
898	4
123,026	3,4880
6	
738,156	

Ans. 36*l.* 18s. 1d. 3½q. nearly.

Examples for practice.

Reduce 7s. 9¾d. to the decimal of a pound. *Ans.* 0,390625.

Reduce 3qrs. 2na. to the decimal of a yard. *Ans.* 0,875.

Find the value of 0,85251*l.* in shillings, pence, &c.

Ans. 17s. 0d. 2½q. nearly.

Reduce 241*l.* 18s. 9d. to federal money. *Ans.* $806,4583 &c.

Find the value of 0,42857 of a month.

Ans. 1w. 4d. 23h. 59′ 56″.

Required the circumference of the earth in English statute miles, a degree being estimated at 57008 toises†.

Ans. 24855,488.

We have given rules for reducing a compound number from one denomination to another, as we shall have frequent occasion in what follows for making these reductions. They are not, however, necessary, except in particular cases, previously to performing the fundamental operations. The several denominations of a compound number may be regarded like the different orders of units in a simple one, that is, the number or numbers of each denomination may be made the subject of a distinct operation, the result of which, being reduced when necessary, may be united to the next, and so on through all the denominations.

† A toise or French fathom is equal to 6 French feet, and a French foot is equal to $12,789\dot{3}$ English inches.

ADDITION OF COMPOUND NUMBERS.

103. THE addition of compound numbers depends on the same principles as that of simple numbers, the object being simply to unite parts of the same denomination, and when a number of these are found, sufficient to form one, or more than one of a higher, these last are retained to be united to others of the same denomination in the given numbers; as in simple addition the tens are carried from one column to the next column on the left. *We must, then, place the compound numbers, that are to be added, in such a manner, that their units, or parts of the same name, may stand under each other; we must then find separately the sum of each column, always recollecting how many parts of each denomination it takes to make one of the next higher.* See the following example in pounds, shillings and pence.

£	s.	d.
984	12	8
38	6	9
1413	14	10
319	18	2
2756	12	5

First, adding together the pence, because they are the parts of the least value, and taking together both the units and tens of this denomination, we find 29; but as 12 pence make a shilling, this sum amounts to 2 shillings and 5 pence; we then write down only the 5 pence, and retain the shillings in order to unite them to the column to which they belong.

Next, we add separately the units and the tens of the next denomination; the first give, by joining to them the 2 shillings reserved from the pence, 22; we write down only the two units and retain the two tens for the next column, the sum of which, by this means, amounts to 5 tens, but as the pound, made up of 20 shillings, contains 2 tens, we obtain the number of pounds resulting from the shillings, by dividing the tens of these last by 2; the quotient is 2, and the remainder 1, which last is written under the column to which it belongs, while the pounds are reserved for the next column on the left; as this column is the last the operation is performed as in simple numbers, and the whole sum is found to be 2756*l.* 12s. 5d.

The method of proving the addition of compound numbers is derived from the same principles as that for simple numbers, and is performed in the same manner, care being taken in passing from one denomination to another, to substitute instead of the decimal ratio, the value of each part in the terms of that, which follows it on the right. Let there be, for example,

£	s.	d.
984	12	8
38	6	9
1413	14	10
319	18	2
2, 56	12	5
1122	22	0

The operation on the pounds is performed according to the rule of article 19; then we change the two pounds into tens of shillings, and obtain 4 of these tens, which, joined to that written under the column, makes 5, from which we subtract the 3 units of this column, and place the remainder, 2, underneath, counting it as tens with regard to the next column. There still remain 2 shillings, which must be reduced to pence; adding the result, 24 pence, to the 5 that are written, we have a total of 29, which must be again obtained by the addition of all the pence, as these are the parts of the lowest denomination in the question. This really happens, and proves the operation to be right.

Examples.

	£	s.	d.	£	s.	d	£	s.	d.
	17	13	4	84	17	$5\frac{1}{2}$	175	10	10
	13	10	2	75	13	$4\frac{1}{4}$	107	13	$11\frac{3}{4}$
	10	17	3	51	17	$8\frac{3}{4}$	89	18	10
	8	8	7	20	10	$10\frac{1}{4}$	75	12	$2\frac{1}{4}$
	3	3	4	17	15	$4\frac{1}{2}$	3	3	$3\frac{3}{4}$
		8	8	10	10	11	1		$\frac{1}{2}$
Sum	54	1	4	261	5	$8\frac{1}{4}$	452	19	$2\frac{1}{4}$
Proof	23	32	0	24	23	20	232	13	0

lb.	oz.	dwt.	gr.
17	3	15	11
13	2	13	13
15	3	14	14
13	10		
12	1		17
		13	14

lb.	oz.	dwt.	gr.
14	10	13	20
13	10	18	21
14	10	10	10
10	1	2	3
1	4	4	4
	1	19	

lb.	oz.	dwt.	gr.
27	10	17	18
17	10	13	13
13	11	13	1
10	1		2
4	4	3	3
2			1

cwt.	qr.	lb.	oz.	dr.
15	2	15	15	15
13	2	17	13	14
12	2	13	14	14
10	1	17	15	
12	1	10		10
10	1	12	1	7

T.	cwt.	qr.	lb.	oz.	dr.
2	17	3	13	8	7
2	13	3	14	8	8
1	16		10		5
2	13			1	7
1	14	1	1	2	2
4	16	1	7	7	5

T.	cwt.	qr.	lb.	oz.	dr.
3	13	2	10	7	7
2	14	1	17	6	6
4	17		14		6
2	13		12	7	7
3	13		10	4	4
5		2	12	8	8

Mls.	fur.	pol.	yd.	ft.	in.
37	3	14	2	1	5
28	4	17	3	2	10
17	4	4	3	1	2
10	5	6	3	1	7
29	2	2	2		3
30			4		2

Mls.	fur.	pol.	yd.	ft.	in.
28	2	13	1	1	4
39	1	17	2	2	10
28	1	14	2	2	
48	1	17	2	2	7
37	1	29			3
2		20		2	1

Mls.	fur.	pol.	yd.	ft.	in.
28	3	7	2		7
30			1		7
27	6	30	2	2	
7	6	20	2	1	
5	2			2	10
	7	10		2	2

SUBTRACTION OF COMPOUND NUMBERS.

104. This operation is performed in the same way as the subtraction of simple numbers, except with regard to the number which it is necessary to borrow from the higher denominations, in order to perform the partial subtractions, when the lower number exceeds the upper. For instance,

	£	s.	d.
from	795	3	0
take	684	17	4
Difference	110	5	8

In performing this example, it is necessary to borrow, from the column of shillings, 1 shilling or 12 pence, in order to effect the subtraction of the lower number, 4, and we have for a remainder 8 pence. There now remain in the upper number of the column of shillings only 2, it is necessary therefore to borrow, from that of pounds, 1 pound or 20 shillings, we thus make it 22, of which, when the lower number, 17, is subtracted, 5 remain; we must now proceed to the column of pounds, remembering to count the upper number less by unity, and finish the operation as in the case of simple numbers.

The method of proving subtraction of compound numbers, like that for simple numbers, consists in adding the difference to the less of the two numbers.

Examples for practice.

	£	s.	d.	£	s.	d.	£	s.	d.
	275	13	4	454	14	$2\frac{3}{4}$	274	14	$2\frac{1}{4}$
	176	16	6	276	17	$5\frac{1}{2}$	85	15	$7\frac{3}{4}$
Rem.	98	16	10	177	16	$9\frac{1}{4}$	188	18	$6\frac{1}{2}$
Proof	275	13	4	454	14	$2\frac{3}{4}$	274	14	$2\frac{1}{4}$

	lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.
	7	3	14	11	27	2	10	20	29	3	14	5
	3	7	15	20	20	3	5	21	20	7	15	7
Rem.												
Proof												

	cwt.	qr.	lb.	oz.	dr.	cwt.	qr.	lb.	oz.	dr.	cwt.	qr.	lb.	oz.	dr.
	5		17	5	9	22	2	13	4	8	21	1	7	6	13
	3	3	21	1	7	20	1	17	6	6	13		8	8	14
Rem.															
Proof															

	Mls.	fur.	pol.	yd.	ft.	in.
	14	3	17	1	2	1
	10	7	30	2		10
Rem.						
Proof						

	Mls.	fur.	pol.	yd.	ft.	in.
	70	7	13	1	1	2
	20		14	2	2	7
Rem.						
Proof						

	Mls.	fur.	pol.	yd.	ft.	in.
	70	3	10			7
	17	3	11	1	1	3
Rem.						
Proof						

	m.	w.	d.	h.	′
	17	2	5	17	26
	10			18	18
Rem.					
Proof					

	m.	w.	d.	h.	′
	37	1		13	1
	15	2		15	14
Rem.					
Proof					

	m.	w.	d.	h.	′
	71				5
	17		5	5	7
Rem.					
Proof					

MULTIPLICATION OF COMPOUND NUMBERS.

105. We have seen, that a number consisting of several denominations may be reduced to a single one, either the lowest or the highest of those contained in it, in which state it admits of being used as an abstract number. But when it is required to find the product of two numbers, one of which only is compound, the simplest method is to consider the multiplication of each denomination of the compound number by the simple factor, as a distinct question, and the several results, thus obtained, will be the total product sought. If it were proposed, for example, to multiply 7*l.* 14s. 7d. 3q. by 9, it may be done thus,

£	s.	d.	q.
7	14	7	3
9	9	9	9
63	126	63	27

and 63*l.* 126s. 63d. 27q. is evidently 9 times the proposed sum, because it is 9 times each of the parts, which compose this sum.

But 27q. is equal to 6d. 3q. and adding the 6d. to the 63d. we have 69d. equal to 5s. 9d. adding the 5s. to the 126s. we obtain 131s. equal to 6*l.* 11s, and lastly, adding the 6*l.* to the 63*l.* we have 69*l.* 11s. 9d. 3q. equal to the above result, and equal to the product of

7*l.* 14s. 7d. 3q. by 9.

Instead of finding the several products first, and then reducing them, we may make the reductions after each multiplication, putting down what remains of this denomination, and carrying forward the quotient, thus obtained, to be united to the next higher product.

Hence, *to multiply two numbers together, one of which is compound, make the compound number the multiplicand and the simple number the multiplier, and beginning with the lowest denomination of the multiplicand, multiply it by the multiplier and divide the product by the number, which it takes to make one of the next superior denomination; putting down the remainder, add the quotient to the product of the next denomination by the multiplier, reduce this sum, putting down the remainder and reserving the quotient, as before, and proceed in this manner through all the denominations to the last, which is to be multiplied like a simple number.*

When the multiplier exceeds 12, that is, when it is so large that it is inconvenient to multiply by the whole at once, the shortest method is to resolve it, if it can be done, into two or more factors, and to multiply first by one and then that product by the other, and so on, as in the following example. Let the two numbers be £4 13s. 3d. and 18.

£	s.	d.
4	13	3
		9
41	19	3
		2
83	18	6

Here we first find 9 times the multiplicand, or £41 19s. 3d. and then take twice this product, which will evidently be twice 9, or 18 times the original multiplicand (82). Instead of multiplying by 9 we might multiply first by 3 and then that product by 3, which would give the same result; also the multiplier 18 might be resolved into 3 and 6, which would give the same product as the above. If we multiply £83 18s. 6d. by 7.

£	s.	d.
83	18	6
		7
587	9	6

we shall have the product of the original multiplicand by 7 times 18 or 126.

If the multiplier were 105, it might be resolved into 7, 3, and 5, and the product be found as above.

But it frequently happens, that the multiplier cannot be resolved in this way into factors. When this is the case, we may take the number nearest to it, which can be so resolved, and find the product of the multiplicand by this number, as already described, and then add or subtract so many times the multiplicand, as this number falls short, or exceeds the given multiplier, and the result will be the product sought. Let there be £1 7s. 8d. to be multiplied by 17.

	£	s.	d.
	1	7	8
			4
	5	10	8
			4
	22	2	8
	1	7	8
Product	£23	10	4

In the first place, I find the product of £1 7s. 8d. by 16, which is £22 2s. 8d. and to this I add once the multiplicand and this sum £23 10s. 4d. is evidently equal to 17 times the multiplicand.

106. It may be observed, that in those cases, where the decrease of value from one denomination to another, is according to the same law throughout, that is, where it takes the same number of a lower denomination to make one of the next higher through all the denominations, the multiplication of one compound number by another may be performed in a manner similar to what takes place with regard to abstract numbers.

This regular gradation is sometimes preserved in the denominations, that succeed to feet in long measure, 1 inch or *prime* being considered as equal to 12 *seconds*, and 1 *second* to 12 *thirds*, and so on, the several denominations after feet being distinguished by one, two, &c. accents, thus,

$$10\text{f. } 4' \quad 5'' \quad 10'''.$$

If it were required to find the product of 2f. 4′ by 3f. 10′, we should proceed as below.

$$\begin{array}{rrr} & 2\text{f.} & 4' \\ & 3 & 10 \\ \hline 1 & 11 & 4 \\ 7 & 0 & \\ \hline 8 & 11 & 4'' \end{array}$$

The 4 inches or primes may be considered with reference to the denomination of feet, as 4 twelfths, or $\frac{4}{12}$, and the 10 inches as $\frac{10}{12}$. the product of which is $\frac{40}{144}$, or $\frac{40}{12}$ of $\frac{1}{12}$, or 40″, which reduced gives 3′ 4″; putting down the 4″, we reserve the 3′ to be added to the product of 2 feet by 10′, or $\frac{10}{12}$, which product is $\frac{20}{12}$ of a foot, to which 3 being added, we have $\frac{23}{12}$f. or 1f. and 11′; next multiplying 4′ or $\frac{4}{12}$ by 3, we have $\frac{12}{12}$ or 1, which added to the product of 2 by 3 gives 7. Taking the sum of these results, we have 8f. 11′ 4″, for the product of 2f. 4′ by 3f. 10′. The method here pursued may be extended to those cases, where there is a greater number of denominations.

Whence, *to multiply one number consisting of feet, primes, seconds, &c. by another of the same kind, having placed the several terms of the multiplier under the corresponding ones of the multiplicand, multiply the whole multiplicand by the several terms of the multiplier successively according to the rule of the last article, placing the first term of each of the partial products under its respective multiplier, and find the sum of the several columns, observing to carry one for every twelve in each part of the operation; then the first number on the left will be feet, and the second primes, and the third seconds, and so on regularly to the last*†.

† The above article relates to what is commonly called *duodecimals.* The operation is ordinarily performed by beginning with the

Examples for practice.

Multiply £1 11s. 6d. 2q.	by 5.	*Ans.*	£7 17s. 8d. 2q.
Multiply 7s. 4d. 3q.	by 24.	*Ans.*	£8 17s. 6d.
Multiply £1 17s. 6d.	by 63.	*Ans.*	£118 2s. 6d.
Multiply 17s. 9d.	by 47.	*Ans.*	£41 14s. 3d.
Multiply £1 2s. 3d.	by 117.	*Ans.*	£130 3s. 3d.

What is the value of 119 yards of cloth at £2 4s. 3d. per yard? *Ans.* £263 5s. 9d.

What is the value of 9cwt. of cheese at £1 11s. 5d. per cwt? *Ans.* £14 2s. 9d.

What is the value of 96 quarters of rye at £1 3s. 4d. per quarter. *Ans.* £112.

What is the weight of 7 hhds. of sugar, each weighing 9 cwt. 3qrs. 12lb. *Ans.* 69. cwt.

In the Lunar circle of 19 years, of 365d. 5h. 48′ 48″ each, how many days, &c.? *Ans.* 6939d. 14h. 27′ 12″.

Multiply 14f. 9′ by 4f. 6′. *Ans.* 66f. 4′ 6″.

Multiply 4f. 7′ 8″ by 9f. 6′. *Ans.* 44f. 0′ 10″.

Required the content of a floor 48f. 6′ long and 24f. 3′ broad. *Ans.* 1176f. 1′ 6″.

What is the number of square feet &c. in a marble slab, whose length is 5f. 7′ and breadth 1f. 10′? *Ans.* 10f. 2′ 10.″

---

highest denomination of the multiplier, and disposing of the several products as in the first example below. The result is evidently the same whichever method is pursued, as may be seen by comparing this example with that of the same question on the right, performed according to the rule in the text. This last arrangement seems to be preferable, as it is more strictly conformable to what takes place in the multiplication of numbers accompanied by decimals.

f.	′	″		
10	4	5		
7	8	1		
72	6	11		
6	10	11	4‴	
	5	2	2	6⁗
79	11	0	6	6

	f.	′	″	
	10	4	5	
	7	8	6	
	5	2	2	6
6	10	11	4	
72	6	11		
79f.	11′	0″	6‴	6⁗

DIVISION OF COMPOUND NUMBERS.

107. A COMPOUND number may be divided by a simple number, by regarding each of the terms of the former, as forming a distinct dividend. If we take the product found in article 105, namely, £63 126s. 63d. 27q. and divide it by the multiplier 9, we shall evidently come back to the multiplicand, £7 14s. 7d. 3q. We arrive at the same result also, by dividing the above sum reduced, or £69 11s. 9d. 3q. for we obtain one 9th of each of the several parts that compose the number, the sum of which must be one 9th of the whole. But si ce, in this case, each term of the dividend is not exactly divisible by the divisor, instead of employing a fraction we reduce what remains, and add it to the next lower denomination, and then divide the sum thus formed, by the divisor. The operation may be seen below.

```
£69  11s.  9d.  3q. | 9
 63                 |£7 14s. 7d. 3q.
 —
  6
 20
 ——
131
  9
 ——
 41
 36
 —
  5
 12
 —
 69
 63
 —
  6
  4
 —
 27
 27
 —
  0
```

Whence, *to divide a number consisting of different denominations by a simple number, divide the highest term of the compound number by the divisor, reduce the remainder to the next lower denomination, adding to it the number of this denomination, and divide the sum by the divisor, reducing the remainder, as before, and proceed in this way through all the denominations to the last, the remainder of which, if there be one, must have its quotient represented in the form of a fraction by placing the divisor under it. The sum of the several quotients, thus obtained, will be the whole quotient required.*

When the divisor is large and can be resolved into two or more simple factors, we may divide first by one of these factors, and then that quotient by another, and so on, and the last quotient will be the same as that which would have been obtained by using the whole divisor in a single operation. Taking the result of the example in the corresponding case of multiplication, we proceed thus,

£83 18s. 6d.	2	
8		
—	£41 19s. 3d.	9
3	36	
2	—	£4 13s. 3d.
—	5	
1	20	
20	—	
—	119	
38	9	
2	—	
—	29	
18	27	
18	—	
—	2	
0	12	
6	—	
6	27	
—	27	
0	—	
	0	

By dividing £83 18s. 6d. by 2, we obtain one half of this sum, which being divided by 9, must give one 9th of one half, or one 18th of the whole. The first operation may be considered as separating the dividend into two equal parts, and the second as distributing each of these into nine equal parts, the number of parts therefore will be 18, and being equal, one of them must be one 18th of the whole.

But when the divisor connot be thus resolved, the operation must be performed by dividing by the whole at once. If the quotient, which we are seeking, were known, by adding it to, or subtracting from it, the dividend a certain number of times, increasing or diminishing the divisor at the same time by as many units, we might change the question into one, whose divisor would admit of being resolved into factors, which would give the same quotient; we should thus preserve the anology which exists between the multiplication and division of compound numbers. But this cannot be done, as it supposes that to be known, which is the object of the operation.

Multiplication and division, where compound numbers are concerned, mutually prove each other, as in the case of simple numbers. This may be seen by comparing the examples, which are given at length to illustrate these rules.

Examples for practice

Divide £821 17s. 9¾d.	by 4.	*Ans.* £205 9s. 5¼d.
Divide £28 2s. 1½d.	by 6.	*Ans.* £4 13s. 8¼d.
Divide £57 3s. 7d.	by 35	*Ans.* £1 12s. 8d.
Divide £23 15s. 7½d.	by 37	*Ans.* 12s. 10¼d.
Divide 1061cwt. 2qrs.	by 28.	*Ans.* 37cwt. 3qrs. 18lb.

Divide 375mls. 2fur. 7pls. 2yds. 1ft. 2in. by 39.

Ans. 9mls. 4fur. 39pls. 2ft. 8in.

If 9 yards of cloth cost £4 3s. 7½d. what is it per yard?

Ans. 9s. 3d. 2q.

If a hogshead of wine cost £33 12s. what is it per gallon?

Ans. 10s. 8d.

If a dozen silver spoons weigh 3lb. 2oz. 13pwt. 12grs. what is the weight of each spoon. *Ans.* 3oz. 4pwt. 11grs.

If a person's income be £150 a year, what is it per day?

Ans. 8s. 2½d. nearly.

A capital of £223 16s. 8d. being divided into 96 shares, what is the value of a share? *Ans.* £2 6s. 7½d.

PROPORTION.

108. We have shown, in the preceding part of this work, the different methods necessary for performing on all numbers, whether whole or fractional, or consisting of different denominations, the four fundamental operations of arithmetic, namely, addition, subtraction, multiplication and division; and all questions relative to numbers ought to be regarded as solved, when, by an attentive examination of the manner in which they are stated, they can be reduced to some one of these operations. Consequently, we might here terminate all that is to be said on arithmetic, for what remains belongs, properly speaking, to the province of algebra. We shall, nevertheless, for the sake of exercising the learner, now resolve some questions which will prepare him for algebraic analysis, and make him acquainted with a very important theory, that of ratios and proportions, which is ordinarily comprehended in arithmetic.

109. *A piece of cloth* 13 *yards long was sold for* 130 *dollars, what will be the price of a piece of the same cloth* 18 *yards long.*

It is plain, that if we knew the price of one yard of the cloth that was sold, we might repeat this price 18 times, and the result would be the price of the piece 18 yards long. Now, since 13 yards cost 130 dollars, one yard must have cost the thirteenth part of 130 dollars, or $\frac{130}{13}$, performing the divison, we find for the result 10 dollars, and multiplying this number by 18, we have 180 dollars for the answer; which is the true cost of the piece 18 yards long.

A courier, who travels always at the same rate, having gone 5 *leagues in* 3 *hours, how many will he go in* 11 *hours?*

Reasoning as in the last example, we see, that the courier goes in one hour $\frac{1}{3}$ of 5 leagues, or $\frac{5}{3}$, and consequently, in 11 hours he will go 11 times as much, or $\frac{5}{3}$ of a league multiplied by 11, or $\frac{55}{3}$, that is 18 leagues and 1 mile.

In how many hours will the courier of the preceding question go 22 *leagues?*

We see, that if we knew the time he would occupy in going one league, we should have only to repeat this number 22 times, and the result would be the number of hours required. Now the courier, requiring 3 hours to go 5 leagues, will require only $\frac{1}{5}$ of the time, $\frac{3}{5}$ of an hour, to go one league; this number, multiplied by 22, gives $\frac{66}{5}$ or 13 hours and $\frac{1}{5}$, that is, 13 hours and 12 minutes.

110. We have discovered the unknown quantities by an analysis of each of the preceding statements, but the known numbers and those required depend upon each other in a manner, that it would be well to examine.

To do this, let us resume the first question, in which it was required to find the price of 18 yards of cloth, of which 13 cost 130 dollars.

It is plain, that the price of this piece would be double, if the number of yards it contained were double that of the first; that if the number of yards were triple, the price would be triple also, and so on; also that for the half or two thirds of the piece we should have to pay only one half or two thirds of the whole price.

According to what is here said, which all those, who understand the meaning of the terms, will readily admit, we see, that if there be two pieces of the same cloth, the price of the second ought to contain that of the first, as many times as the length of the second contains the length of the first, and this circumstance is stated in saying, that the prices are in proportion to the lengths, or have the same relation to each other as the lengths.

This example will serve to establish the meaning of several terms which frequently occur.

111. The *relation* of the lengths is the number, whether whole or fractional, which denotes how many times one of the lengths contains the other. If the first piece had 4 yards and the second 8, the relation, or ratio, of the former to the latter would be 2, because 8 contains 4 twice. In the above example, the first piece had 13 yards and the second 18, the ratio of the former to the latter is then $\frac{18}{13}$, or $1\frac{5}{13}$. In general, *the relation* or *ratio of two numbers is the quotient arising from dividing one by the other.*

As the prices have the same relation to each other, that the lengths have, 180 divided by 130 must give $\frac{18}{13}$ for a quotient, which is the case; for in reducing $\frac{180}{130}$ to its most simple terms, we get $\frac{18}{13}$.

The four numbers, 13, 18, 130, 180, written in this order, are then such, that the second contains the first as many times as the fourth contains the third, and thus they form what is called a proportion.

We see also, that *a proportion is the combination of two equal ratios.*

We may observe, in this connexion, that a relation is not changed by multiplying each of its terms by the same number; and this is plain, because a relation, being nothing but the quotient of a division, may always be expressed in a fractional form. Thus the relation $\frac{18}{13}$ is the same as $\frac{180}{130}$.

The same considerations apply also to the second example. The courier, who went 5 leagues in 3 hours, would go twice as far in double that time, three times as far in triple that time; thus 11 hours, the time spent by the courier in going 18 leagues and $\frac{1}{3}$, or $\frac{55}{3}$ of a league, ought to contain 3 hours, the time required in going 5 leagues, as often as $\frac{55}{3}$ contains 5.

The four numbers 5, $\frac{55}{3}$, 3, 11, are then in proportion; and in reality if we divide $\frac{55}{3}$ by 5, we get $\frac{55}{15}$, a result equivalent to $\frac{11}{3}$. It will now be easy to recognize all the cases, where there may be a proportion between the four numbers.

112. To denote that there is a proportion between the numbers 13, 18, 130, and 180, they are written thus,

$$13 : 18 :: 130 : 180,$$

which is read 13 *is to* 18 *as* 130 *is to* 180; that is, 13 is the same part of 18 that 130 is of 180, or that 13 is contained in 18 as many times as 130 is in 180, or lastly, that the relation of 18 to 13 is the same as that of 180 to 130.

The first term of a relation is called the *antecedent*, and the second the *consequent*. In a proportion there are two *antecedents* and two *consequents*, viz. the antecedent of the first relation and that of the second; the consequent of the first relation and that of the second. In the proportion 13 : 18 : : 130 : 180, the antecedents are 13, 130; the consequents 18 and 180.

We shall in future take the consequent for the numerator, and the antecedent for the denominator of the fraction which expresses the relation.

113. To ascertain that there is a proportion between the four numbers 13, 18, 130, and 180, we must see if the fractions $\frac{18}{13}$ and $\frac{180}{130}$ be equal, and to do this, we reduce the second to its most simple terms; but this verification may also be made by considering, that if, as is supposed by the nature of proportion, the two fractions $\frac{18}{13}$ and $\frac{180}{130}$ be equal, it follows that, by reducing them to the same denominator, the numerator of the one will become equal to that of the other, and that, consequently, 18 multiplied by 130 will give the same product as 180 by 13. This is actually the case, and the reasoning by which it is shown, being independent of the particular values of the numbers, proves, that, *if four numbers be in proportion, the product of the first and last, or of the two extremes, is equal to the product of the second and third, or of the two means.*

We see at the same time, that, if the four given numbers were not in proportion, they would not have the abovementioned property; for the fraction, which expresses the first ratio, not being equivalent to that which expresses the second, the numerator of the one will not be equal to that of the other, when they are reduced to a common denominator.

114. The first consequence, naturally drawn from what has been said, is, that the order of the terms of a proportion may be changed, provided they be so placed, that the product of the extremes shall be equal to that of the means. In the proportion 13 : 18 : : 130 : 180, the following arrangements may be made;

$$
\begin{array}{r}
13 : 18 :: 130 : 180 \\
13 : 130 :: 18 : 180 \\
180 : 130 :: 18 : 13 \\
180 : 18 :: 130 : 13 \\
18 : 13 :: 180 : 130 \\
18 : 180 :: 13 : 130 \\
130 : 13 :: 180 : 18 \\
130 : 180 :: 13 : 18
\end{array}
$$

for in each one of these, the product of the extremes is formed of the same factors, and the product of the means of the same factors. The second arrangement, in which the means have changed places with each other, is one of those that most frequently occur.*

115. This change shows that we may either multiply or divide the two antecedents, or the two consequents, by the same number, without destroying the proportion. For this change makes the two antecedents to constitute the first relation, and the two consequents, the second. If, for instance, 55 : 21 : : 165 : 63, changing the places of the means we should have,

55 : 165 : : 21 : 63 ;

we might now divide the terms, which form the first relation, by 5, (111) which would give 11 : 33 : : 21 : 63, changing again the places of the means, we should have 11 : 21 : : 33 : 63, a proportion which is true in itself, and which does not differ from the given proportion, except in having had its two antecedents divided by 5.

116. Since the product of the extremes is equal to that of the means, one product may be taken for the other, and, as in dividing the product of the extremes, by one extreme, we must necessarily find the other as the quotient, *consequently, in dividing by one extreme the product of the means, we shall find the other extreme.* For the same reason, *if we divide the product of the extremes by one of the means, we shall find the other mean.*

---

* It may be observed, that the proportion 13 : 130 : : 18 : 180 might have been at once presented under this form, according to the solution of the question in article 109; for the value of a yard of cloth may be ascertained in two ways, namely, by dividing the price of the piece of 13 yards by 13, or by dividing the price of 18 yards by 18 ; it follows then that the price of the first must contain 13 as many times as the price of the second contains 18; we shall then have 13 : 130 : : 18 : 180. We may reason in the same manner with respect to the 2nd question in the article above referred to, as well as with respect to all others of the like kind, and thence derive proportions; but the method adopted in article 109 seemed preferable, because it leads us to compare together numbers of the same denomination, whilst by the others we compare prices, which are sums of money, with yards, which are measures of length; and this cannot be done without reducing them both to abstract numbers.

We can then find any one term of a proportion, when we know the other three. for the term sought must be either one of the extremes or one of the means.

The question of article (109) may be resolved by one of these rules. Thus, when we have perceived that the prices of the two pieces are in the proportion of the number of yards contained in each, we write the proportion in this manner,

$$13 : 18 : : 130 : x,$$

putting the letter x instead of the required price of 18 yards, and we find the price, which is one of the extremes, by multiplying together the two means, 18 and 130, which makes 2340. and dividing this product by the known extreme, 13; we obtain, for the result, 180.

The operation, by which, when any three terms of a proportion are given, we find the fourth, is called the *Rule of Three.* Writers on arithmetic have distinguished it into several kinds, but this is unnecessary, when the nature of proportion and the enunciation of the question are well understood; as a few examples will sufficiently show.

117. A person having travelled 217,5 miles in 9 days; it is asked, how long he will be in travelling 423,9 miles, he being supposed to travel at the same rate?

In this question the unknown quantity is the number of days, which ought to contain the 9 days spent in going 217,5 miles, as many times as 423,9 contains 217,5; we thus get the following proportion;

$217{,}5 : 423{,}9 : : \overset{\text{days}}{9} : x$, and we find for x, 17,54 nearly.

118. All the difficulty in these questions consists in the manner of stating the proportion. The following rules will be sufficient to guide the learner in all cases.

Among the four numbers which constitute a proportion, there are two of the same kind, and two others also of the same kind, but different from the first two. In the preceding example, two of the terms are miles, and the other two; days.

First, then, it is necessary to distinguish the two terms of each kind, and when this is done, we shall necessarily have the quotient of the greatest term of the second kind by the smallest of the same kind, equal to the quotient of the greatest term of the first kind by the smallest of the same kind, which will give us this proportion,

the smaller term of the first kind
is
to the larger of the same kind
as
the smaller term of the second kind
is
to the larger of this kind.

In the preceding example this rule immediately gives,

$$217,5 : 423,9 :: 9 : x$$

for the unknown term ought to be greater than 9, since a greater number of days will be necessary to complete a longer journey.

119. If it were required to find how many days it would take 27 men to perform a piece of work, which 15 men, working at the same rate, would do in 18 days; we see that the days should be less in proportion as the number of men is greater, and reciprocally. There is still a proportion in this case, but the order of the terms is inverted; for, if the number of workmen in the second set were triple of that in the first, they would require only one third of the time. The first number of days then would contain the second as many times, as the second number of workmen would contain the first. This order of the terms being the reverse of that assigned to them by the enunciation of the question, we say, that the number of workmen is in the *inverse ratio* of the number of days. If we compare the two first, and the two last, in the order in which they present themselves, the ratio of the former will be 3, or $\frac{3}{1}$, and that of the latter $\frac{1}{3}$, which is the same as the preceding with the terms inverted.

It is evident, indeed, that we invert a ratio by inverting the terms of the fraction, which expresses it, since we make the antecedent take the place of the consequent, and the consequent that of the antecedent. $\frac{3}{2}$ or $2 : 3$ is the inverse of $\frac{2}{3}$ or $3 : 2$.

The mode of proceeding in such cases may be rendered very simple; for we have only to take the numbers denoting the two sets of workmen, for the quantities of the first kind, and the numbers denoting the days, for those of the second, and to place the one and the other in the order of their magnitude; proceeding thus, we have the following proportion,

$$15 : 27 :: x : 18,$$

from which we immediately find x equal to 10.

Recapitulating the remarks already given, we have the following rule; *make the number which is of the same kind with the answer the third term, and the two remaining ones the first and second, putting the greater or the less first, according as the third is greater or less than the term sought; then the fourth term will be found by multiplying together the second and third, and dividing the product by the first.*

120. 1st. A man placed 3575 dollars at interest at the rate of 5 per cent. yearly; it is asked what will be the interest of this sum at the end of one year?

The expression. 5 per cent. interest, means, that for a sum of one hundred dollars, 5 dollars is allowed at the end of a year; if then, we take the two principals for the quantities of the first kind, and the interest for those of the second, we shall have,

$$100 : 3575 :: 5 : x,$$

a proportion which may be reduced to $20 : 3575 :: 1 : x$, according to the observation in article 115; then dividing the two terms of the first relation by 5, we shall have $4 : 715 :: 1 : x$, whence x is equal to $\frac{715}{4}$, or \$178,75 cts.

We may also resolve this question by considering that 5 is $\frac{1}{20}$ of 100, and that consequently we shall obtain the interest of any sum put out at this rate by taking the twentieth part of this sum. Now $\frac{1}{20}$ of \$3575 is \$178,75; a result which agrees with the one before found.

2d. A merchant gives his note for \$800,00 payable in a year; the note is sold to a broker, who advances the money for it eight months before the time of payment; how much ought the broker to give?

As the broker advances, from his own funds, a sum, which is not to be replaced till the expiration of 8 months, it is proper that he should be allowed interest for his money during this time.

Let the interest for a year be 6 per cent. the interest for 8 months will be $\frac{8}{12}$, or $\frac{2}{3}$, of 6, or 4; a sum then of 100 dollars, lent for 8 months, must be entitled to 4 dollars interest; that is, he who borrows it ought to return \$104. By considering the \$800, as a sum so returned for what is advanced by the broker, we shall have this proportion, $104 : 100 :: 800 : x$, whence we get \$769,23† for the value of x, that is, for the sum the broker ought to give.*

Questions for practice.

What is the value of a cwt. of sugar at $5\frac{1}{2}$d. per lb.?

Ans. 2*l.* 11s. 4d.

What is the value of a chaldron of coals at $11\frac{1}{2}$d. per bushel?

Ans 1*l.* 14s. 6d.

What is the value of a pipe of wine at $10\frac{1}{2}$d. per pint?

Ans. 44*l.* 2s.

At 3*l.* 9s. per cwt. what is the value of a pack of wool, weighing 2cwt. 2qrs. 13lb. *Ans.* 9*l.* 6d. $\frac{12}{112}$.

What is the value of $1\frac{1}{2}$cwt. of coffee at $5\frac{1}{2}$d. per ounce?

Ans. 61*l.* 12s.

Bought 3 casks of raisins, each weighing 2cwt. 2qrs. 25lb. what will they come to at 2*l.* 1s. 8d. per cwt?

Ans. 17*l.* $4\frac{3}{4}$d. $\frac{32}{112}$.

What is the value of 2qrs. 1nl. of velvet at 19s. $8\frac{1}{2}$d. per English ell? *Ans.* 8s. $10\frac{1}{4}$d. $\frac{14}{20}$.

Bought 12 pockets of hops, each weighing 1cwt. 2qrs. 17lb.; what do they come to at 4*l.* 1s. 4d. per cwt.?

Ans. 80*l.* 12s. $1\frac{1}{2}$d. $\frac{96}{112}$.

What is the tax upon 745*l.* 14s. 8d. at 3s. 6d. in the pound?

Ans. 130*l.* 10s. $0\frac{3}{4}$d $\frac{48}{240}$.

---

† A sum, thus advanced, is called the *present worth* of the sum due at the expiration of the proposed time.

* The operation by which we find what ought to be given for a sum of money, when the time of payment is anticipated, belongs to what is called *Discount*. There are several ways of calculating discount, but the above is the most exact, as it has regard merely to simple interest.

If $\frac{3}{4}$ of a yard of velvet cost 7s. 3d. how many yards can I buy for 15*l*. 15s. 6d. ? *Ans.* $28\frac{1}{2}$ yards.

If an ingot of gold, weighing 9lb. 9oz. 12dwt. be worth 411*l*. 12s. what is that per grain ? *Ans.* $1\frac{3}{4}$d.

How many quarters of corn can I buy for 140 dollars at 4s. per bushel ? *Ans.* 26qrs. 2bu.

Bought 4 bales of cloth, each containing 6 pieces, and each piece 27 yards, at 16*l*. 4s. per piece ; what is the value of the whole, and the rate per yard ?

Ans. 388*l*. 16s. at 12s. per yard.

If an ounce of silver be worth 5s. 6d. what is the price of a tankard, that weighs 1lb. 10oz. 10dwt. 4gr.

Ans. 6*l*. 3s. $9\frac{1}{2}$d. $\frac{96}{480}$.

What is the half year's rent of 547 acres of land at 15s. 6d. per acre ? *Ans.* 211*l*. 19s. 3d.

At $1,75 per week, how many months' board can I have for 100*l*. ? *Ans.* 47. m. 2w. $\frac{60}{126}$.

Bought 1000 Flemish ells of cloth for 90*l*. how must I sell it per ell in Boston to gain 10*l*. by the whole ? *Ans.* 3s. 4d.

If a gentleman's income is 1750 dollars a year, and he spends 19s. 7d. per day, how much will he have saved at the year's end ? *Ans.* 167*l*. 12s. 1d.

What is the value of 172 pigs of lead, each weighing 3cwt. 2qrs. $17\frac{1}{2}$lb. at 8*l*. 17s. 6d. per fother of $19\frac{1}{2}$cwt.

Ans. 286*l*. 4s. $4\frac{1}{2}$d.

The rents of a whole parish amount to 1750*l*. and a tax is granted of 32*l*. 16s. 6d. what is that in the pound ?

Ans. $4\frac{1}{2}$d. $\frac{2880}{420000}$.

If keeping of one horse be $11\frac{1}{2}$d. per day, what will be that of 11 horses for a year ? *Ans.* 192*l*. 7s. $8\frac{1}{2}$d.

A person breaking owes in all 1490*l*. 5s. 10d. and has in money, goods, and recoverable debts, 784*l*. 17s. 4d. if these things be delivered to his creditors, what will they get on the pound ? *Ans.* 10s. $6\frac{1}{4}$d. $\frac{20933}{33767}$.

What must 40s. pay towards a tax, when 652*l*. 13s. 4d. is assessed at 83*l*. 12s. 4d. ? *Ans.* 5s. $1\frac{1}{4}$d. $\frac{15376}{153[illegible]}$.

Bought 3 tuns of oil for 151*l*. 14s. 85 gallons of which being damaged, I desire to know how I may sell the remainder per gallon, so as neither to gain nor lose by the bargain?

Ans. 4s. $6\frac{1}{4}$d. $\frac{25}{671}$.

What quantity of water must I add to a pipe of mountain wine, valued at 33*l.* to reduce the first cost to 4s. 6d. per gallon?

Ans. $20\frac{2}{3}$ gallons.

If 15 ells of stuff, $\frac{3}{4}$ yard wide, cost 37s. 6d. what will 40 ells of the same stuff cost, being a yard wide? *Ans.* 6*l.* 13s. 4d.

Shipped for Barbadoes 500 pairs of stockings at 3s. 6d. per pair, and 1650 yards of baize at 1s. 3d. per yard, and have received in return 348 gallons of rum at 6s. 8d. per gallon, and 750lb. of indigo at 1s. 4d. per lb. what remains due upon my adventure? *Ans.* 24*l.* 12s. 6d.

If 100 workmen can finish a piece of work in 12 days, how many are sufficient to do the same in 3 days? *Ans.* 400 men.

How many yards of matting, 2ft. 6in. broad, will cover a floor, that is 27ft. long, and 20ft. broad. *Ans.* 72 yards.

How many yards of cloth, 3qrs. wide, are equal in measure to 30 yards 5qrs. wide? *Ans.* 50 yards.

A borrowed of his friend B 250*l.* for 7 months, promising to do him the like kindness; sometime after B had occasion for 300*l.* how long may he keep it to receive full amends for the favor? *Ans.* 5 months and 25 days.

If, when the price of a bushel of wheat is 6s. 3d. the penny loaf weigh 9oz. what ought it to weigh when wheat is at 8s. $2\frac{1}{2}$d. per bushel? *Ans.* 6oz. 13dr.

If $4\frac{1}{2}$cwt. can be carried 36 miles for 35 shillings, how many pounds can be carried 20 miles for the same money?

Ans. 907lb. $\frac{4}{20}$.

How many yards of canvass, that is an ell wide, will line 20 yards of say, that is 3qrs. wide? *Ans.* 12yds.

If 30 men can perform a piece of work in 11 days, how many men will accomplish another piece of work, 4 times as big, in a fifth part of the time? *Ans.* 600.

A wall that is to be built to the height of 27 feet, was raised 9 feet by 12 men in 6 days; how many men must be employed to finish the wall in 4 days at the same rate of working?

Ans. 36.

If $\frac{5}{7}$oz. cost $\frac{11}{12}$l. what will 1oz. cost ? *Ans.* 1*l.* 5s. 8d.

If $\frac{3}{16}$ of a ship cost 273*l.* 2s. 6d. what is $\frac{5}{32}$ of her worth ? *Ans.* 227*l.* 12s. 1d.

At $1\frac{1}{2}$*l.* per cwt. what does $3\frac{1}{3}$lb. come to ? *Ans.* $10\frac{5}{7}$d.

If $\frac{5}{8}$ of a gallon cost $\frac{5}{8}$*l.* what will $\frac{5}{9}$ of a tun cost ? *Ans.* 140*l.*

A person, having $\frac{3}{5}$ of a coal mine, sells $\frac{3}{4}$ of his share for 171*l.* what is the whole mine worth ? *Ans.* 380*l.*

If, when the days are $13\frac{5}{8}$ hours long, a traveller perform his journey in $35\frac{1}{2}$ days ; in how many days will he perform the same journey, when the days are $11\frac{9}{10}$ hours long ?

Ans. $40\frac{6}{9}\frac{15}{52}$ days.

A regiment of soldiers, consisting of 976 men, are to be new clothed, each coat to contain $2\frac{1}{2}$ yards of cloth, that is $1\frac{5}{8}$yd. wide, and to be lined with shalloon, $\frac{7}{8}$yd. wide ; how many yards of shalloon will line them ? *Ans.* 4531yds. 1qr. $2\frac{6}{7}$nl.

COMPOUND PROPORTION.

121. Proportion is also applied to questions, in which the relation of the quantity required, to the given quantity of the same kind, depends upon several circumstances, combined together ; it is then called *Compound Proportion,* or *Double Rule of Three.* See some examples.

It is required to find how many leagues a person would go in 17 days, travelling 10 hours a day, when he is known to have travelled 112 leagues, in 29 days, employing only 7 hours a day.

This question may be resolved in two ways, we will first give the one that leads to Compound Proportion.

In each case, the number of leagues passed over depends upon two circumstances, namely, the number of days the man travels, and the number of hours he travels in each day.

We will not at first consider this latter circumstance, but suppose the number of hours be the same in each case ; the question then will be ; *a person in* 29 *days travels* 112 *leagues, how many will he travel in* 17 *days?* This will furnish the following proportion ;

$$29 : 17 :: 112 : x.$$

The fourth term will be equal to 112 multiplied by 17 and divided by 29, or $\frac{1904}{29}$ leagues.

Now, to take into consideration the number of hours, we must say, if a person travelling 7 hours a day, for a certain number of days, has travelled $\frac{1904}{29}$ leagues, how far will he go in the same time, if he travel 10 hours a day? This will lead to the following proportion,

$$\overset{\text{h.}}{7} : \overset{\text{h.}}{10} : : \overset{\text{l.}}{\tfrac{1904}{29}} : x,$$

which gives for the fourth term, or answer, 93,793 leagues nearly.

The question may also be resolved by observing, that 29 days travelling, at 7 hours a day, is equal to 203 hours travelling; and that 17 days, at 10 hours a day, amounts to 170 hours; the problem then is reduced to this proportion,

$$203 : 170 : : 112 : x,$$

by which we find the distance he ought to travel in 170 hours, according to what he performed in 203 hours.

We see, by the first mode of resolving the question, that 112 leagues has to the fourth term, or answer, the same proportion, that 29 days has to 17, and that 7 hours has to 10; stating this in the form of a proportion, we have

$$\left.\begin{array}{r} \overset{\text{d.}}{29} : \overset{\text{d.}}{17} \\ \overset{\text{h.}}{7} : \overset{\text{h.}}{10} \end{array}\right\} : : \overset{\text{lea.}}{112} : x,$$

by which it appears, that 112 is to be multiplied by both 17 and 10, and to be divided by both 29 and 7, that is, 112 is to be multiplied by the product of 17 by 10, and divided by the product of 29 by 7, which is the same as the second method of resolving the question.

122. Again, if 9 labourers, working 8 hours a day, have spent 24 days in digging a ditch 65 yards long, 13 wide, and 5 deep, how many days will it take 71 labourers of equal ability, working 11 hours a day, to dig a ditch 327 yards long, 18 broad, and 7 deep?

Here is a question very complicated in appearance, but which may be resolved by proportion.

If all the conditions of these two cases were alike, except the number of men and the number of days, the question would consist only in finding in how many days 71 men would perform the work, which 9 men have done in 24 days ; we should have then,

$$71 : 9 :: 24 : x,$$

but instead of calculating the number of days, we will only indicate, as in article 82, the numbers to be multiplied together, and place as a denominator those by which they are to be divided ; we thus have for x days,

$$\frac{24 \text{ by } 9}{71}.$$

But as the first labourers worked only 8 hours a day, while the others worked 11, the number of days required by the latter will be less in proportion, which will give

$$11 : 8 :: \frac{24 \text{ by } 9}{71} : x;$$

whence we conclude that the number of days, in this case, is

$$\frac{24 \text{ by } 9 \text{ by } 8}{71 \text{ by } 11}.$$

This number is that of the days necessary for 71 labourers, working 11 hours a day, to dig the first ditch.

The ditches being of unequal length, as many more days will be necessary, as the second is longer than the first, thus we shall have

$$65 : 327 :: \frac{24 \text{ by } 9 \text{ by } 8}{71 \text{ by } 11} : x,$$

and the number of days, this new circumstance being considered, will be

$$\frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327}{71 \text{ by } 11 \text{ by } 65}.$$

Taking into consideration the breadths, which are not alike, we have

$$13 : 18 :: \frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327}{71 \text{ by } 11 \text{ by } 65} : x,$$

and, consequently, the number of days required changes to

$$\frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327 \text{ by } 18}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13}.$$

Lastly, the depths being different, we have

$$5 : 7 :: \frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327 \text{ by } 18}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13} : x,$$

and the number of days, resulting from the concurrence of all these circumstances, is

$$\frac{24 \text{ by } 9 \text{ by } 8 \text{ by } 327 \text{ by } 18 \text{ by } 7}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13 \text{ by } 5}.$$

Performing the multiplications and divisions, we get the answer required, 21 days $\frac{1902831}{3299725}$.

123. This number is equal to 24 multiplied by the fractional quantity

$$\frac{9 \text{ by } 8 \text{ by } 327 \text{ by } 18 \text{ by } 7}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13 \text{ by } 5};$$

but this last quantity, which expresses the relation of the number of days given, to the number of days required, is itself the product of the following fractions;

$$\frac{9}{71}, \frac{8}{11}, \frac{327}{65}, \frac{18}{13}, \frac{7}{5}.$$

Now, going back to the denominations given to these numbers in the statement of the question, we see that these fractions are the ratios of the men and the hours, of the lengths, the breadths and the depths of the two ditches; hence it follows, that the ratio of the number of days given, to the number of days sought, is equal to the product of all the ratios, which result from a comparison of the terms relating to each circumstance of the question.

This may be resolved in a simple manner by first finding the value of each of these ratios; for, by multiplying together the fractions that express them, we form a fraction which represents the ratio of the quantity required to the given quantity of the same kind.

This fraction, which will be the product of all the ratios in the question, will have for its numerator the product of all the antecedents, and for its denominator, that of all the consequents. A ratio resulting, in this manner, from the multiplication of several others, is called a *compound ratio.*

By means of the fractional expression

$$\frac{9 \text{ by } 8 \text{ by } 327 \text{ by } 18 \text{ by } 7}{71 \text{ by } 11 \text{ by } 65 \text{ by } 13 \text{ by } 5},$$

and the given number of days, 24, we make the following proportion, 71 by 11 by 65 by 13 by 5 : 9 by 8 by 327 by 18 by 7 : : 24 : x, which may also be represented in this manner, as was the preceding example.

$$\left.\begin{array}{r} 71 : \quad 9 \\ 11 : \quad 8 \\ 65 : 327 \\ 13 : \quad 18 \\ 5 : \quad 7 \end{array}\right\} : : 24 : x.$$

Our remarks may be summed up in this rule; *Make the number, which is of the same kind with the required answer, the third term; and of the remaining numbers, take any two that are of the same kind, and place one for a first term and the other for a second term, according to the directions in simple proportion; then any other two of the same kind, and so on, till all are used; lastly, multiply the third term by the product of the second terms, and divide the result by the product of the first terms, and the quotient will be the fourth term, or answer required.*

Examples for practice.

If 100*l.* in one year gain 5*l.* interest, what will be the interest of 750*l.* for 7 years? *Ans.* 262*l.* 10s.

What principal will gain 262*l.* 10s. in 7 years, at 5*l.* per cent. per annum? *Ans.* 750*l.*

If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles? *Ans.* $9\frac{63}{65}$ days.

If 120 bushels of corn can serve 14 horses 56 days; how many days will 94 bushels serve 6 horses? *Ans.* $102\frac{16}{45}$ days.

If 7oz. 5dwt. of bread be bought at $4\frac{3}{4}$d. when corn is at 4s. 2d. per bushel, what weight of it may be bought for 1s. 2d. when the price per bushel is 5s. 6d.? *Ans.* 1lb. 4oz. $3\frac{479}{627}$dwt.

If the transportation of 13cwt. 1qr. 72 miles be 2*l.* 10s. 6d. what will be the transportation of 7cwt. 3qrs. 112 miles?

Ans. 2*l.* 5s. 11d. $1\frac{77}{159}$q.

A wall, to be built to the height of 27 feet, was raised to the height of 9 feet by 12 men in 6 days; how many men must be employed to finish the wall in 4 days, at the same rate of working? *Ans.* 36 men.

If a regiment of soldiers, consisting of 939 men, consume 351 quarters of wheat in 7 months; how many soldiers will consume 1464 quarters in 5 months, at that rate? *Ans.* $5483\frac{23}{195}$.

If 248 men, in 5 days of 11 hours each, dig a trench 230 yards long, 3 wide and 2 deep; in how many days of 9 hours long, will 24 men dig a trench of 420 yards long, 5 wide and 3 deep? *Ans.* $288\frac{59}{207}$.

FELLOWSHIP.

124. THE object of this rule is to divide a number into parts, which shall have a given relation to each other; we shall see in the following example its origin, and whence it has its name.

Three merchants formed a company for the purpose of trade; the first advanced 25000 dollars, the second 18000, and the third 42000; after some time they separated, and wished to divide the joint profit, which amounted to 57225 dollars; how much ought each one to have?

To resolve this question we must consider, that each man's gain ought to have the same relation to the whole gain, as the money he advanced has to the whole sum advanced; for he, who furnishes a half or third of this sum, ought, plainly, to have a half or third of the profit. In the present example, the whole sum being 85000 dollars, the particular sums will be respectively $\frac{25000}{85000}$, $\frac{18000}{85000}$, $\frac{42000}{85000}$ of it; and if we multiply these fractions by the whole gain, 57225, we shall obtain the gain belonging to each man. It is moreover evident, that the sum of the parts will be equal to the whole gain, because the sum of the above fractions, having its numerator equal to its denominator, is necessarily an unit.

We have therefore these proportions;

\$ \$ \$
85000 : 25000 : : 57225 : to the first man's gain,
85000 : 18000 : : 57225 : to the second man's gain,
85000 : 42000 : : 57225 : to the third man's gain,

which may be enunciated thus;

The whole sum advanced : to each man's particular sum : : the whole gain : to each man's particular gain.

By simplifying the first ratio of each of these proportions we have

$$
\begin{aligned}
&\qquad\quad \$\\
&85 : 25 : : 57225 : \text{to the gain of the } 1^{\text{st.}} \text{ or } \$16830\tfrac{75}{85},\\
&85 : 18 : : 57225 : \text{to the gain of the } 2^{\text{d.}} \text{ or } \$12118\tfrac{21}{85},\\
&85 : 42 : : 57225 : \text{to the gain of the } 3^{\text{d.}} \text{ or } \$28275\tfrac{75}{85}.
\end{aligned}
$$

If all the sums advanced had been equal, the operation would have been reduced to dividing the whole gain by the number of sums advanced; we may reduce the question to this in the present case, by supposing the whole sum, \$85000, divided into 85 partial sums, or stocks of \$1000 each, the gain of each of these sums will evidently be the $85^{\text{th.}}$ part of the whole gain; and nothing remains to be done, except to multiply this part severally by 25, 18, and 42, considering the sums 25000, 18000, and 42000 as the amounts of 25 shares, 18 shares, and 42 shares.

In commercial language the money advanced is called the *capital* or *stock*, and the gain to be divided, the *dividend*.

The following question is very similar to that just resolved.

125. It is required to divide an estate of 67250 dollars among 3 heirs, in such a manner, that the share of the second shall be $\frac{2}{5}$ of that of the first, and the share of the third $\frac{7}{8}$ of that of the second.

It is plain that the share of the third, compared with that of the first, will be $\frac{7}{8}$ of $\frac{2}{5}$ of it, or $\frac{7}{20}$; then the three parts required will be to each other in the proportion of the numbers 1, $\frac{2}{5}$ and $\frac{7}{20}$. Reducing these to a common denominator, we find them $\frac{20}{20}$, $\frac{8}{20}$, and $\frac{7}{20}$, and have the three numbers 20, 8, and 7, which are proportional to the first; but as their sum is 35, it is plain, that if we take three parts expressed by the fractions, $\frac{20}{35}$, $\frac{8}{35}$, and $\frac{7}{35}$, they will be in the required proportion. The question will then be resolved by taking $\frac{20}{35}$, then $\frac{8}{35}$, and then $\frac{7}{35}$ of 67250 dollars, which will give the sums due to the heirs, according to the manner prescribed, namely;

$$\$38428\tfrac{20}{35},\ \$15371\tfrac{15}{35},\ \text{and } \$13450.$$

126. Again, there are two fountains, the first of which will fill a certain reservoir in $2\frac{1}{2}$ hours, and the second will fill the same reservoir in $3\frac{3}{4}$ hours; how much time will be required to

fill the reservoir, by means of both fountains running at the same time?

We must first ascertain what part of the reservoir will be filled by the first fountain in any given time, an hour for instance. It is evident that, if we take the content of the reservoir for unity, we have only to divide 1 by $2\frac{1}{2}$, or $\frac{5}{2}$, which gives us $\frac{2}{5}$ for the part filled in one hour by the first fountain. In the same manner, dividing 1 by $3\frac{3}{4}$, or $\frac{15}{4}$, we obtain $\frac{4}{15}$ for the part of the reservoir filled in an hour by the second fountain; consequently, the two fountains, flowing together for an hour, will fill $\frac{2}{5}$ added to $\frac{4}{15}$, or $\frac{10}{15}$ of the reservoir. If we now divide 1, or the content of the reservoir, by $\frac{10}{15}$, we shall obtain the number of hours necessary for filling it at this rate; and shall find it to be $\frac{15}{10}$ or an hour and a half.

Authors who have written upon arithmetic, have multiplied and varied these questions in many ways, and have reduced to rules the processes which serve to resolve them; but this multiplication of precepts is, at least, useless, because a question of the kind we have been considering may always be solved with facility by one who perceives what follows from the enunciation; especially when he can avail himself of the aid of algebra; we shall therefore proceed to another subject.

Besides the proportions composed of four numbers, one of the two first of which contains the other as many times as the corresponding one of the two last contains the other; it has been usual to consider as such the assemblage of four numbers, such as 2, 7, 9, 14, of which one of the two first exceeds the other as much as the corresponding one of the two last exceeds the other.

These numbers, which may be called *equidifferent*, possess a remarkable property, analogous to that of proportion, for the sum of the extreme terms, 2 and 14, is equal to the sum of the means, 7 and 9*.

---

* The ancients kept the theory of proportions very distinct from the operations of arithmetic. Euclid gives this theory in the fifth book of his elements, and as he applies the proportions to lines, it is apparent, that we thence derive the name of *geometrical proportion*:

To show this property to be general, we must observe, that the second term is equal to the first increased by the difference, and that the fourth is equal to the third increased by the difference; hence it follows, that the sum of the extremes, composed of the first and fourth terms, must be equal to the first increased by the third increased by the difference. Also, that the sum of the means, composed of the second and third terms, must be equal to the first increased by the difference increased by the third term; these two sums, being composed of the same parts, must consequently be equal.

We have gone on the supposition, that the second and fourth terms were greater than the first and third; but the contrary may be the case, as in the four numbers 8, 5, 15, 12; the second term will be equal to the first decreased by the difference, and the fourth will be equal to the third decreased by the difference. By changing the word *increased* into *decreased*, in the preceding reasoning, it will be proved that, in the present case, the sum of the extremes is equal to that of the means.

We shall not pursue this theory of equidifferent numbers further, because, at present, it can be of no use.

Questions for practice.

A and B have gained by trading $182. A put into stock $300 and B $400; what is each person's share of the profit?

Ans. A $78 and B $104.

---

and that the name of *arithmetical proportion* was given to an assemblage of equidifferent numbers, which were not treated of till a much later period. These names are very exceptionable; the word *proportion* has a determinate meaning, which is not at all applicable to equidifferent numbers. Besides, the proportion called *geometrical* is not less arithmetical than that which exclusively possesses the latter name. M. Lagrange, in his Lectures at the Normal school, has proposed a more correct phraseology, and I have thought proper to follow his example.

Equidifference, or the assemblage of four equidifferent numbers, or arithmetical proportion, is written thus; 2 . 7 : 9 . 14.

Among English writers the following form is used;

2 . . 7 : : 9 . . 14.

Divide \$120 between three persons, so that their shares shall be to each other as 1, 2, and 3, respectively.

Ans. \$20, \$40, and \$60.

Three persons make a joint stock. A put in \$185,66, B \$98,50, and C \$76,85; they trade and gain \$222; what is each person's share of the gain?

Ans. A \$104,17$\frac{83}{36101}$, B \$60,57$\frac{6243}{36101}$, and C 47,25$\frac{29775}{36101}$.

Three merchants, A, B, and C, freight a ship with 340 tuns of wine; A loaded 110 tuns, B 97, and C the rest. In a storm the seamen were obliged to throw 85 tuns overboard; how much must each sustain of the loss?

Ans. A 27$\frac{1}{2}$, B 24$\frac{1}{4}$, and C 33$\frac{1}{4}$.

A ship worth \$860 being entirely lost, of which $\frac{1}{8}$ belonged to A, $\frac{1}{4}$ to B, and the rest to C; what loss will each sustain, supposing \$500 of her to be insured?

Ans. A \$45, B \$90, and C \$225.

A bankrupt is indebted to A \$277,33, to B \$305,17, to C \$152, and to D \$105. His estate is worth only \$677,50; how must it be divided? *Ans.* A \$223,81$\frac{2580}{8395}$, B \$246,28$\frac{615}{8395}$, C \$122,66$\frac{6930}{8395}$, and D \$84,73$\frac{6665}{8395}$.

A and B, venturing equal sums of money, clear by joint trade \$154. By agreement A was to have 8 per cent. because he spent his time in the execution of the project, and B was to have only 5 per cent.; what was A allowed for his trouble?

Ans. \$35,53$\frac{11}{13}$.

Three graziers hired a piece of land for \$60,50. A put in 5 sheep for 4$\frac{1}{2}$ months, B put in 8 for 5 months, and C put in 9 for 6$\frac{1}{2}$ months; how much must each pay of the rent?

Ans. A \$11,25, B \$20, and C \$29,25.

Two merchants enter into partnership for 18 months; A put into stock at first \$200, and at the end of 8 months he put in \$100 more; B put in at first \$550, and at the end of 4 months took out \$140. Now at the expiration of the time they find they have gained \$526; what is each man's just share?

Ans. A's \$192,95$\frac{70}{1254}$, B's \$333,04$\frac{1184}{1254}$.

A, with a capital of \$1000, began trade January 1, 1776, and meeting with success in business he took in B a partner, with a capital of \$1500 on the first of March following. Three months after that, they admit C as a third partner, who brought into stock $2800, and after trading together till the first of the next year, they find the gain, since A commenced business, to be $1776,50. How must this be divided among the partners?

Ans. A's $457,40$\frac{364}{466}$
B's 571,83$\frac{222}{466}$
C's 747,19$\frac{346}{466}$

ALLIGATION.

128. We shall not omit the rule of alligation, the object of which is to find the mean value of several things of the same kind, of different values; the following examples will sufficiently illustrate it.

A wine merchant bought several kinds of wine, namely;

130 bottles which cost him 10 cents each,	
75	at 15
231	at 12
27	at 20

he afterwards mixed them together; it is required to ascertain the cost of a bottle of the mixture. It will be easily perceived, that we have only to find the whole cost of the mixture and the number of bottles it contains, and then to divide the first sum by the second, to obtain the price required.

Now, the 130 bottles	at	10 cents	cost 1300 cents
75	at	15	cost 1125,
231	at	12	cost 2772,
27	at	20	cost 540,
then 463 bottles	cost		5737 cents.

5737 divided by 463 gives 12,39 cents, the price of a bottle of the mixture.

The preceding rule is also used for finding a mean of different results, given by experiment or observation, which do not agree with each other. If, for instance, it were required to know the distance between two points considerably removed from each other, and it had been measured; whatever care might have been used in doing this, there would always be a little uncertainty in the result, on account of the errors inevitably committed by the manner of placing the measures one after the other.

We will suppose that the operation has been repeated several times, in order to obtain the distance exactly, and that twice it has been found 3794yds. 2ft. that three other measurements have given 3795yds. 1ft. and a third 3793yds. As these numbers are not alike, it is evident that some must be wrong, and perhaps all. To obtain the means of diminishing the error, we reason thus; if the true distance had been obtained by each measurement, the sum of the results would be equal to six times that distance, and it is plain that this would also be the case, if some of the results obtained were too little, and others too great, so that the increase, produced by the addition of the excesses, should make up for what the less measurements wanted of the true value. We should then, in this last case, obtain the true value by dividing the sum of the results by the number of them.

This case is too peculiar to occur frequently, but it almost always happens, that the errors on one side destroy a part of those on the other, and the remainder, being equally divided among the results, becomes smaller according as the number of results is greater.

According to these considerations we shall proceed as follows;

	yds.	ft.			ft.
We take twice	3794	2	or	7589	1
3 times	3795	1	or	11386	0
once	3793		or	3793	
6 results, giving in all				22768	1.

Dividing 22768yds. 1ft. by 6, we find the mean value of the required distance to be 3794yds. 2ft.

129. Questions sometimes occur, which are to be solved by a method, the reverse of that above given. It may be required, for example, to find what quantity of two different ingredients it would take to make a mixture of a certain value. It is evident, that if the value of the mixture required exceeds that of one of the ingredients just as much as it falls short of that of the other, we should take equal quantities of each to make the compound.

So also, if the value of the mixture exceeded that of one twice as much as it fell short of that of the other, the proportion of the ingredients would be as one half to one. To mix wine at \$2 per gallon with wine at \$3, so that the compound shall be worth \$2,50, we should take equal quantities, or quantities in the proportion of 1 to 1. If the mixture were required to be worth \$2,66$\frac{2}{3}$, the quantities would be in the proportion of $\frac{1}{2}$ to 1, or of $\frac{1}{66\frac{2}{3}}$ to $\frac{1}{33\frac{1}{3}}$, and generally, the nearer the mixture rate is to that of one of the ingredients, the greater must be the quantity of this ingredient with respect to the other, and the reverse; hence, *To find the proportion of two ingredients of a given value, necessary to constitute a compound of a required value, make the difference between the value of each ingredient and that of the compound the denominator of a fraction, whose numerator is one, and these fractions will express the proportion required;* and being reduced to a common denominator, the numerators will express the same proportion, or show what quantity of each ingredient is to be taken to make the required compound.

When the compound is limited to a certain quantity, the proportion of the ingredients, corresponding to it, may be found by saying; as the whole quantity, found as above, is to the quantity required, so is each part, as obtained by the rule, to the required quantity of each.

Let it be required, for example, to mix wine at 5s. per gallon and 8s. per gallon, in such quantities that there may be 60 gallons worth 6s. per gallon. The difference between 6s. and 5s. is 1, and between 6s. and 8s. is 2, giving for the required quantities the ratio of $\frac{1}{1}$ to $\frac{1}{2}$, or 2 to 1; thus, taking x equal to the quantity at 5s. and y equal to the quantity at 8s. we have these proportions; $3 : 60 :: 2 : x$, and $3 : 60 :: 1 : y$, giving, for the answer, 40 gallons at 5s. and 20 gallons at 8s. per gallon.

Also, when one of the ingredients is limited, we may say; as the quantity of the ingredient found as above, is to the required quantity of the same, so is the quantity of the other ingredient to the proportional part required.

For example, I would know how many gallons of water at 0s. per gallon, I must mix with thirty gallons of wine at 6s. per gallon, so that the compound may be worth 5s. per gallon. First, the difference between 0s. and 5s. is 5; and the difference between 6s. and 5s. is 1; the quantity of water therefore will be to that of the wine, as $\frac{1}{5}$ to $\frac{1}{1}$, or as 1 to 5. Then, from this ratio, we institute the proportion, $5 : 30 :: 1 : x$, which gives 6, for the number of gallons required.

As we have found the proportion of two ingredients necessary to form a compound of a required value, so also we may consider either of these in connexion with a third, with a fourth, and so on, thus making a compound of any required value, consisting of any number whatever of simple ingredients. The two ingredients used, however, must always be, one of a greater and the other of a less value, than that of the compound required.

A grocer would mix teas at 12s. and 10s. with 40lbs. at 4s. per pound, in such proportions that the composition shall be worth 8s. per lb. If he mix only two kinds, the one at 4s. and the other at 10s. their quantities will be in the ratio of $\frac{1}{4}$ to $\frac{1}{2}$, or $1 : 2$; and if he mix the tea at 4s. also with that at 12s. their ratio will be that of $\frac{1}{4}$ to $\frac{1}{4}$, or of 1 to 1. Adding together the proportions of the ingredient, which is taken with each of the others, we find the several quantities, at 4s. 10s. and 12s. to be as 2, 2, and 1. And taking x for the number of lbs. at 10s. and y for the quantity at 12s. we have the following proportions;

$$2 : 40 :: 2 : x \text{; and } 2 : 40 :: 1 : y \text{;}$$

giving, for the answer, 40lb. at 10s. and 20lb. at 12s. per pound.

The problems of the two last articles are generally distinguished by the names of alligation *medial*, and alligation *alternate*. A full explanation of the latter belongs properly to algebra.

Examples.

A composition being made of 5lb. of tea at 7s. per pound, 9lb. at 8s. 6d. per pound, and $14\frac{1}{2}$lb. at 5s. 10d. per pound; what is a pound of it worth? *Ans.* 6s. $10\frac{1}{2}$d.

How much gold, of 15, of 17, and of 22 carats† fine, must be mixed with 5oz. of 18 carats fine, so that the composition may be 20 carats fine? *Ans.* 5oz. of 15 carats fine, 5 of 17, and 25 of 22.

† A carat is a twenty fourth part; 22 carats fine means $\frac{22}{24}$ of pure metal. A carat is also divided into four parts, called grains of a carat.

Miscellaneous Questions for practice.

What number, added to the thirty-first part of 3813, will make the sum 200? *Ans.* 77.

The remainder of a division is 325, the quotient 467, and the divisor is 43 more than the sum of both; what is the dividend? *Ans.* 390270.

Two persons depart from the same place at the same time; the one travels 30, the other 35 miles a day; how far are they distant at the end of 7 days, if they travel both the same road; and how far, if they travel in contrary directions? *Ans.* 35, and 455 miles.

A tradesman increased his estate annually by 100*l.* more than $\frac{1}{4}$ part of it, and at the end of 4 years found that his estate amounted to 10342*l.* 3s. 9d. What had he at first? *Ans.* 4000*l.*

Divide 1200 acres of land among A, B, and C, so that B may have 100 more than A, and C 64 more than B. *Ans.* A 312, B 412, and C 476.

Divide 1000 crowns; give A 120 more, and B 95 less, than C. *Ans.* 445, B 230, C 325.

What sum of money will amount to 132*l.* 16s. 3d. in 15 months, at 5 per cent. per annum, simple interest? *Ans.* 125*l.*

A father divided his fortune among his sons, giving A 4 as often as B 3, and C 5 as often as B 6; what was the whole legacy, supposing A's share 5000*l.*? *Ans.* 11875*l.*

If 1000 men, besieged in a town with provisions for 5 weeks, each man being allowed 16oz. a day, were reinforced with 500 men more. On hearing, that they cannot be relieved till the end of 8 weeks, how many ounces a day must each man have, that the provision may last that time? *Ans.* $6\frac{2}{3}$.

What number is that, to which if $\frac{2}{7}$ of $\frac{5}{9}$ be added, the sum will be 1? *Ans.* $\frac{53}{63}$.

A father dying left his son a fortune, $\frac{1}{4}$ of which he spent in 8 months; $\frac{3}{7}$ of the remainder lasted him twelve months longer; after which he had only 410*l.* left. What did his father bequeath him? *Ans.* 956*l.* 13s. 4d.

A guardian paid his ward 3500*l.* for 2500*l.* which he had in his hands 8 years. What rate of interest did he allow him?
Ans. 5 per cent.

A person, being asked the hour of the day, said, the time past noon is equal to $\frac{4}{5}$ of the time till midnight. What was the time?
Ans. 20min. past 5.

A person, looking on his watch, was asked, what was the time of the day; he answered, it is between 4 and 5; but a more particular answer being required, he said, that the hour and minute hands were then exactly together. What was the time?
Ans. $21\frac{9}{11}$ min. past 4.

With 12 gallons of Canary, at 6s. 4d. a gallon, I mixed 18 gallons of white wine, at 4s. 10d. a gallon, and 12 gallons of cider, at 6s. 1d. a gallon. At what rate must I sell a quart of this composition, so as to clear 10 per cent.? *Ans.* 1s. $3\frac{5}{7}$d.

What length must be cut off a board, $8\frac{3}{8}$ inches broad, to contain a square foot; or as much as 12 inches in length and 12 in breadth? *Ans.* $17\frac{13}{67}$in.

What difference is there between the interest of 350*l.* at 4 per cent. for 8 years, and the discount of the same sum, at the same rate, and for the same time? *Ans.* 27*l.* $3\frac{1}{33}$s.

A father devised $\frac{7}{18}$ of his estate to one of his sons, and $\frac{7}{18}$ of the residue to another, and the surplus to his relict for life; the children's legacies were found to be 257*l.* 3s. 4d. different. What money did he leave for the widow? *Ans.* 635*l.* $10\frac{30}{49}$d.

What number is that, from which if you take $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder add $\frac{7}{16}$ of $\frac{1}{20}$, the sum will be 10? *Ans.* $10\frac{191}{2240}$.

A man dying left his wife in expectation that a child would be afterward added to the surviving family; and, making his will, ordered, that if the child were a son, $\frac{2}{3}$ of his estate should belong to him, and the remainder to his mother; but if it were a daughter, he appointed the mother $\frac{2}{3}$, and the child the remainder. But it happened, that the addition was both a son and a daughter, by which the mother lost in equity 2400*l.* more than if it had been only a daughter. What would have been her dowry, had she had only a son?* *Ans.* 2100*l.*

* If she had had a son, she would have have had one, as often as the son two parts of the estate: if a daughter, then two as often as the latter had one part. But, since [illegible] divide the estate among each

A young hare starts 40 rods before a grey-hound, and is not perceived by him till she has been up 40 seconds; she scuds away at the rate of 10 miles an hour, and the dog, on view, makes after her at the rate of 18. How long will the course continue, and what will be the length of it from the place, where the dog set out? *Ans.* $60\frac{5}{22}$ seconds, and 535 yards run.

A reservoir for water has two cocks to supply it; by the first alone it may be filled in 40 minutes, by the second in 50 minutes, and it has a discharging cock, by which it may, when full, be emptied in 25 minutes. Now these three cocks being all left open, the influx and efflux of the water being always at the same rate, in what time would the cistern be filled?

Ans. 3 hours 20 minutes.

A sets out from London for Lincoln precisely at the time, when B at Lincoln sets out for London, distant 100 miles; after 7 hours they met on the road, and it then appeared, that A had ridden $1\frac{1}{2}$ mile an hour more than B. At what rate an hour did each of them travel? *Ans.* A $7\frac{25}{28}$, B $6\frac{11}{28}$ miles.

What part of 3 pence is a third part of 2 pence? *Ans.* $\frac{2}{9}$.

A has by him $1\frac{1}{2}$cwt. of tea, the prime cost of which was 96*l.* sterling. Now interest being at 5 per cent. it is required to find how he must rate it per pound to B, so that by taking his negotiable note, payable at 3 months, he may clear 20 guineas by the bargain? *Ans.* 14s. $1\frac{13}{56}$d. sterling.

There is an island 73 miles in circumference, and 3 footmen all start together to travel the same way about it; A goes 5 miles a day, B 8, and C 10; when will they all come together again? *Ans.* 73 days.

A man, being asked how many sheep he had in his drove, said, if he had as many more, half as many more, and 7 sheep and a half, he should have 20; how many had he? *Ans.* 5.

A person left 40s. to 4 poor widows, A, B, C, and D; to A he left $\frac{1}{3}$, to B $\frac{1}{4}$, to C $\frac{1}{5}$, and to D $\frac{1}{6}$, desiring the whole might be distributed accordingly; what is the proper share of each?

Ans. A's share 14s. $\frac{16}{38}$d. B's 10s. $6\frac{12}{38}$d. C's 8s. $5\frac{2}{38}$d. D's 7s. $\frac{8}{38}$d.

the daughters' = 1, the mothers' = 2, & cons. the sons' = 4. Theref
they will have respectively $\frac{1}{7}$, $\frac{2}{7}$, $\frac{4}{7}$, of the whole estate. Now
mother had only a daughter she would have $\frac{2}{3}$ of the whole estate; and
then ~~state~~ $\frac{2}{4}$; and further in consequence of having both; she lost 24

A general, disposing of his army into a square, finds he has 284 soldiers over and above; but increasing each side with one soldier, he wants 25 to fill up the square; how many soldiers had he? *Ans.* 24000.

There is a prize of 212*l.* 14s. 7d. to be divided among a captain, 4 men, and a boy; the captain is to have a share and a half; the men each a share, and the boy $\frac{1}{3}$ of a share; what ought each person to have?

Ans. The captain 54*l.* 14s. $\frac{3}{7}$d. each man 36*l.* 9s. $4\frac{2}{7}$d. and the boy 12*l.* 3s. $1\frac{3}{7}$d.

A cistern, containing 60 gallons of water, has 3 unequal cocks for discharging it; the greatest cock will empty it in one hour, the second in 2 hours, and the third in 3; in what time will it be emptied, if they all run together? *Ans.* $32\frac{8}{11}$ minutes.

In an orchard of fruit trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ pears, $\frac{1}{6}$ plums, and 50 of them cherries: how many trees are there in all? *Ans.* 600.

A can do a piece of work alone in 10 days, and B in 13; if both be set about it together, in what time will it be finished?

Ans. $5\frac{15}{23}$ days.

A, B, and C are to share 100000*l.* in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, respectively; but C's part being lost by his death, it is required to divide the whole sum properly between the other two.

Ans. A's part is $57142\frac{6}{7}$*l.* and B's $42857\frac{1}{7}$*l.*

APPENDIX,

CONTAINING TABLES OF VARIOUS WEIGHTS AND MEASURES.

New French Weights and Measures.

The weights and measures in common use are liable to great uncertainty and inconvenience. There being no fixed standard at hand, by which their accuracy can be ascertained, a great variety of measures, bearing the same name, has obtained in different countries. The foot, for instance, is used to stand for about a hundred different established lengths. The several denominations of weights and measures are also arbitrary, and occasion most of the trouble and perplexity, that learners meet with in mercantile arithmetic.

To remedy these evils, the French government adopted a new system of weights and measures, the several denominations of which proceed in a decimal ratio, and all referrible to a common permanent standard, established by nature, and accessible at all places on the earth. This standard is a meridian of the earth, which it was thought expedient to divide into 40 million parts. One of these parts is assumed as the unit of length, and the basis of the whole system. This they called a *metre.* It is equal to about $39\frac{1}{3}$ English inches, of which submultiples and multiples being taken, the various denominations of length are formed.

			Eng. Inch. Dec.
A millimetre *is* the	1000th part of a metre		,03937
A centimetre the	100th part of a metre		,39371
A decimetre the	10th part of a metre		3,93710
A METRE			39,37100
A decametre	10	metres	393,71000
A hecatometre	100	metres	3937,10000
A chiliometre	1000	metres	39371,00000
A myriometre	10000	metres	393710,00000
A grade or degree of the meridian equal to 100000 metres, or 100th part of the quadrant			3937100,00000

	Mls.	Fur.	Yds.	Ft.	In.Dc.
The decametre is	0	0	10	2	9,7
The hecatometre	0	0	109	1	1
The chiliometre	0	4	213	1	10,2
The myriometre	6	1	156	0	6
The grade or decimal degree of the meridian	62	1	23	2	8

Measures of Capacity.

A cube, whose side is one tenth of a metre, that is, a cubic decimetre, constitutes the unit of measures of capacity. It is called the *litre,* and contains 61,028 cubic inches.

		Eng. Cub. In. Dec.
A millilitre or 1000th part of a litre		,06103
A centilitre	100th of a litre	,61028
A decilitre	10th of a litre	6,10280
A *litre,* a cubic decimetre		61,02800
A dècalitre	10 litres	610,28000
A hecatolitre	1000 litres	6102,80000
A chiliolitre	10000 litres	61028,00000
A myriolitre	100000 litres	610280,00000

The English pint, wine measure, contains 28,875 cubic inches. The litre therefore is 2 pints and nearly one eighth of a pint.

Hence,

A decalitre is equal to 2 gal. $64\frac{44}{231}$ cubic inches.
A hecatolitre 26 gal. $4\frac{44}{231}$ cubic inches.
A chiliolitre 264 gal. $\frac{44}{231}$ cubic inches.

Weights.

The unit of weight is the *gramme.* It is the weight of a quantity of pure water, equal to a cubic centimetre, and is equal to 15,444 grains Troy.

		Gr. Dec.
A milligramme is 1000th part of a gramme		0,0154
A centigramme	100th of a gramme	0,1544
A decigramme	10th of a gramme	1,5444
A *gramme,* a cubic centimetre		15,4440
A decagramme	10 grammes	154,4402
A hecatogramme	100 grammes	1544,4023

A chiliogramme	1000	grammes	15444,0234
A myriogramme	10000	grammes	154440,2344

A gramme being equal to 15,444 grains Troy.

A decagramme 6dwt. 10,44gr. equal to 5,65 drams Avoirdupois.

		lb.	oz.	dr.	
A hecatogramme	equal to	0	3	8,5	avoird.
A chilogramme		2	3	5	avoird.
A myriogramme		22	1	15	avoird.

100 miriogrammes make a tun, wanting 32lb. 8oz.

Land Measure.

The unit is the *are*, which is a square decametre, equal to 3,95 perches. The deciare is a tenth of an are, the centiare is 100th of an are, and equal to a square metre. The milliare is 1000th of an are. The decare is equal to 10 ares; the hecatare to 100 ares, and equal to 2 acres 1 rood 35,4 perches English. The chilare is equal to 1000 ares, the myriare to 10000 ares.

For fire-wood the *stere* is the unit of measure. It is equal to a cubic metre, containing 35,3171 cubic feet English. The decestere is the tenth of a stere.

The quadrant of the circle generally is divided like the fourth part of the meridian, into 100 degrees, each degree into 100 minutes, and each minute into 100 seconds, &c. so that the same number, which expresses a portion of the meridian, indicates also its length, which is a great convenience in navigation.

The coin also is comprehended in this system, and made to conform to their unit of weight. The weight of the *franc*, of which one tenth is alloy, is fixed at 5 grammes; its tenth part is called *décime*, its hundredth part *centime*.

The divisions of time, soon after the adoption of the above, underwent a similar alteration.

The year was made to consist of 12 months of 30 days each, and the excess of 5 days in common and 6 in leap years was considered as belonging to no month. Each month was divided into three parts, called decades. The day was divided into 10 hours, each hour into 100 minutes, and each minute into 100 seconds. This new calendar was adopted in 1793; in 1805 it was abolished, and the old calendar restored. The weights and measures are still in use, and will probably prevail throughout the continent of Europe. They are recommended to the attention of every civilized country ; and their general adoption would prove of no small importance to the scientific, as well as the commercial world.

Scripture Long Measure.

		Eng. Feet.	In. Dec.
4†	Digit	0	0,912
3	Palm	0	3,648
2	Span	0	10,944
4	Cubit	1	9,888
$1\frac{1}{2}$	Fathom	7	3,552
$1\frac{1}{3}$	Ezekiel's reed	10	11,328
10	Arabian pole	14	7,104
	Scoenus, measuring line	145	1,104

N. B. There was another span used in the East, equal to $\frac{1}{4}$th of a cubit.

Grecian Long Measure reduced to English.

		Eng. paces.	Feet.	In. Dec.
4	Dactylis, Digit	0	0	$0,7554\frac{11}{16}$
$2\frac{1}{2}$	Doron, Dochme, Palesta,	0	0	$3,0218\frac{3}{4}$
$1\frac{1}{10}$	Lichas	0	0	$7,5546\frac{7}{8}$
$1\frac{1}{11}$	Orthodoron	0	0	$8,3101\frac{9}{16}$
$1\frac{1}{3}$	Spithame	0	0	$9,0656\frac{1}{4}$
$1\frac{1}{8}$	Pous, foot	0	1	0,0875
$1\frac{1}{5}$	Pygme, cubit	0	1	$1,5984\frac{3}{8}$
$1\frac{1}{5}$	Pygon	0	1	$3,109\frac{3}{8}$
4	Pecus, cubit larger	0	1	6,13125
100	Orgya, pace	0	6	0,525
8	Stadium, Aulus } furlong	100	4	4,5
	Million, Mile	805	5	0

N. B. Two sorts of long measures were used in Greece, viz. the Olympic and the Pythic. The former was used in Peloponnesus, Attica, Sicily, and the Greek cities in Italy. The latter was used in Thssaly, Illyria, Phocis, and Thrace.

† These numbers show how many of each denomination it takes to make one of the next following.

The Olympic foot, properly called the Greek foot, according to

Dr. Hutton, contains	12,108 English inches,
Folkes,	12,072
Cavallo,	12,084

The Pythic foot, called also natural foot, according to

Hutton, contains	9,768
Paucton,	9,731

Hence it appears, that the Olympic stadium is 201½ English yards nearly; and the Pythic or Delphic stadium, 162½ yards nearly; and the other measures in proportion.

The Phyleterian foot is the Pythic cubit, or 1½ Pythic foot. The Macedonian foot was 13,92 English inches; and Sicilian foot of Archimedes, 8,76 English inches.

Jewish Long or Itinerary Measure.

		Eng. Miles.	Paces.	Feet. Dec.
400	Cubit	0	0	1,824
5	Stadium	0	145	4,6
2	Sabbath day's journey	0	729	3,0
3	Eastern mile	1	403	1,0
8	Parasang	4	153	3,0
	A day's journey	33	172	4,0

Roman Long Measures reduced to English.

		Eng. Paces.	Feet.	In. Dec.
$1\frac{1}{3}$	Digitus traversus	0	0	$0{,}725\frac{3}{4}$
3	Uncia, or Inch	0	0	0,967
4	Palma minor	0	0	2,901
$1\frac{1}{4}$	Pes, or Foot	0	0	11,604
$1\frac{1}{5}$	Palmipes	0	1	2,505
$1\frac{2}{3}$	Cubitus	0	1	5,406
2	Gradus	0	2	5,501
125	Passus	0	4	10,02
8	Stadium	120	4	4,5
	Milliare	967	0	0

N. B. The Roman measures began with 6 scrupula = 1 sicilicum; 8 scrupula = 1 duellum; 1½ duellum = 1 seminaria; and 18 scrupula = 1 digitus. Two passus were equal to 1 decempeda.

Attic Dry Measures reduced to English.

		Pecks.	Gall.	Pints.	Sol. In.
10	Cochliarion	0	0	0	0,276$\frac{7}{26}$
$1\frac{1}{2}$	Cyathus	0	0	0	2,763$\frac{1}{2}$
4	Oxybaphon	0	0	0	4,144$\frac{3}{4}$
2	Cotylus	0	0	0	16,579
$1\frac{1}{2}$	Xestes, sextary	0	0	0	33,158
48	Chœnix	0	0	1	15,705$\frac{1}{4}$
	Medimnus	4	0	6	3,501

Attic Measures of Capacity for Liquids, reduced to English Wine Measure.

		Gal.	Pints.	Sol. In. Dec.
2	Cochliarion	0	$\frac{1}{120}$	0,0356$\frac{5}{12}$
$1\frac{1}{4}$	Cheme	0	$\frac{1}{60}$	0,0719$\frac{5}{6}$
2	Myston	0	$\frac{1}{48}$	0,089$\frac{11}{48}$
2	Concha	0	$\frac{1}{24}$	0,178$\frac{11}{24}$
$1\frac{1}{2}$	Cyathus	0	$\frac{1}{12}$	0,356$\frac{11}{21}$
4	Oxybathon	0	$\frac{1}{8}$	0,535$\frac{3}{8}$
2	Cotylus	0	$\frac{1}{2}$	2,141$\frac{1}{2}$
6	Xestes, sextary	0	1	4,283
12	Chous, congius	0	6	25,698
	Metretes, amphora	10	2	19,626

Others reckon 6 choi = 1 amphoreus, and 2 amphorei = 1 keramion or metretes. The keramion is stated by Paucton to have been equal to 35 French pints, or $8\frac{2}{3}$ English gallons, and the other measures in proportion.

Measures of Capacity for Liquids, reduced to English Wine Measure.

		Gal.	Pints.	Sol. In. Dec.
4	Ligula	0	$\frac{1}{43}$	0,117$\frac{5}{12}$
$1\frac{1}{2}$	Cyathus	0	$\frac{1}{12}$	0,469$\frac{2}{3}$
2	Acetabulum	0	$\frac{1}{8}$	0,704$\frac{1}{2}$
2	Quartarius	0	$\frac{1}{4}$	1,409
2	Hemina	0	$\frac{1}{2}$	2,818
6	Sextarius	0	1	5,636
4	Congius	0	7	4,942
2	Urna	3	$4\frac{1}{2}$	5,33
20	Amphora	7	1	10,66
	Culeus	143	3	11,095

Jewish dry Measures reduced to English.

		Pecks.	Gal.	Pints.	Sol. Inch.
20	Gachal	0	0	$01\frac{17}{20}$	0,031
$1\frac{4}{5}$	Cab	0	0	$2\frac{5}{6}$	0,073
$3\frac{1}{3}$	Gomor	0	0	$5\frac{1}{10}$	1,211
3	Seah	1	0	1	4,036
5	Epha	3	0	3	12,107
2	Letteeh	16	0	0	26,500
	Chomer, coron	32	0	1	18,969

Jewish Measures of Capacity for Liquids, reduced to English Wine Measure.

		Gal.	Pints.	Sol. Inch.
$1\frac{1}{3}$	Caph	0	$\frac{5}{8}$	0,177
4	Log	0	$\frac{5}{6}$	0,211
3	Cab	0	$3\frac{1}{3}$	0,844
2	Hin	1	2	2,533
3	Seah	2	4	5,067
10	Bath, epha	7	4	15,2
	Coron, chomer	75	5	7,625

Ancient Roman Land Measure.

100 Square Roman feet	= 1 Scrupulum of land
4 Scrupula	= 1 Sextulus
$1\frac{1}{5}$ Sextulus	= 1 Actus
6 Sextuli or 5 Actus	= 1 Uncia of land
6 Unciæ	= 1 Square Actus
2 Square Actus	= 1 Jugerum
2 Jugera	= 1 Heredium
100 Heredia	= 1 Centuria

N. B. If we take the Roman foot at 11,6 English inches, the Roman jugerum was 5980 English square yards, or 1 acre $37\frac{1}{2}$ perches.

Roman Dry Measures reduced to English.

		Peck.	Gal.	Pint.	Sol. In. De.
4	Ligula	0	0	$0\frac{1}{48}$	0,01
$1\frac{1}{2}$	Cyathus	0	0	$0\frac{1}{12}$	0,04
4	Acetabulum	0	0	$0\frac{1}{8}$	0,06
2	Hemina or Trutta	0	0	$0\frac{1}{2}$	0,24
8	Sextarius	0	0	1	0,48
2	Semi d.	0	1	0	3,84
	Medius	1	0	0	7,68

Table of the principal Gold and Silver Coins now current, containing their Weight, Fineness, Pure Contents, Current Value, and Intrinsic Value in Sterling, according to the Mint Price of England.

	Names of the Coins.	Weight.	Fineness		Pure Contents.	Current Value.	Value in Sterling.			Dolls.
	Gold Coins.	grs.	car.	grs.	grs.		l.	s.	d.	
Austrian Dominions,	Souverain, single	85,50	22		78,37	6 florins 40 creutzers		13	10	3,074
	Ducat Kremnitz *or* Hungarian	53,85	23	3	53,29	4 florins 30 creutzers		9	$5\frac{5}{4}$	2,097
Bavaria,	Carolin d'or	150,32	18	$2\frac{5}{6}$	117,18	10 florins 42 creutzers	1	0	9	4,611
	Max d'or	100,21	18	$2\frac{5}{6}$	78,12	7 florins 3 creutzers		13	10	3,076
Brunswick,	Carl d'or	102,36	21	3	92,76	5 rix dollars		16	$5\frac{1}{4}$	3,653
Bern,	Ducat	54,31	23	2	53,18	7 livres 4 sous		9	5	2,093
Denmark,	Ducat current	48,21	21	$0\frac{1}{3}$	42,35	12 marks Danish		7	6	1,667
East Indies,	Mohur, *or* gold rupee	176,50	23		169,15	15 silver rupees	1	6	$11\frac{1}{4}$	6,653
	Star pagoda	52,75	19	2	42,86	$3\frac{3}{4}$ silver rupees		7	7	1,685
England,	Guinea	129,44	22		118,65	21 shillings	1	1	0	4,667
	Half guinea	64,72	22		59,32	$10\frac{1}{2}$ shillings		10	6	2,333
	Seven shilling piece	43,15	22		39,55	7 shillings		7	0	1,556
Flanders,	See Austrian Dominions									
France,	Louis d'or, old, (coined before 1786)	125,51	21	$2\frac{1}{3}$	113,09	24 livres		19	$11\frac{3}{4}$	4,44
	Louis d'or, new, (coined since 1786)	117,66	21	$2\frac{1}{2}$	106,02	24 livres		18	$9\frac{1}{4}$	4,171
	Napoleon, or piece of 40 francs, (new coins)	199,25	21	$0\frac{4}{5}$	179,33	40 francs	1	11	$8\frac{3}{4}$	7,051
Geneva,	Pistole	87,08	22		79,82	10 livres		14	$1\frac{1}{2}$	3,139
Genoa,	Sequin	53,90	23	$3\frac{1}{2}$	53,62	13 lire 10 soldi		9	$5\frac{3}{4}$	2,106
	Genovina d'oro	434,20	21	$3\frac{1}{2}$	396,74	100 lire	3	10	$2\frac{1}{2}$	15,597
	New piece of 96 lire	390,	21	$3\frac{1}{4}$	354,45	96 lire	3	2	9	13,945
Germany,	Ducat *ad legem Imperii*	53,85	23	$2\frac{1}{3}$	53,10	varies in different places		9	$4\frac{3}{4}$	2,088
Hamburg,	See Germany									
Hanover,	George d'or	103,03	21	3	93,37	5 rix dollars		16	$6\frac{1}{4}$	3,671
	Gold gulden	50,11	18		37,58	2 rix dollars		6	8	1,481

INTRODUCTION

TO THE

ELEMENTS OF ALGEBRA.

AN

INTRODUCTION

TO THE

ELEMENTS OF ALGEBRA,

DESIGNED FOR THE USE OF THOSE

WHO ARE ACQUAINTED ONLY WITH THE FIRST PRINCIPLES

OF

ARITHMETIC.

SELECTED FROM THE ALGEBRA OF EULER.

Second Edition.

CAMBRIDGE, N. ENG.

PRINTED BY HILLIARD AND METCALF,

At the University Press.

SOLD BY W. HILLIARD, CAMBRIDGE, AND BY CUMMINGS AND HILLIARD,

NO. 1 CORNHILL, BOSTON.

1821.

ADVERTISEMENT.

None but those who are just entering upon the study of Mathematics need to be informed of the high character of Euler's Algebra. It has been allowed to hold the very first place among elementary works upon this subject. The author was a man of genius. He did not, like most writers, compile from others. He wrote from his own reflections. He simplified and improved what was known, and added much that was new. He is particularly distinguished for the clearness and comprehensiveness of his views. He seems to have the subject of which he treats present to his mind in all its relations and bearings before he begins to write. The parts of it are arranged in the most admirable order. Each step is introduced by the preceding, and leads to that which follows, and the whole taken together constitutes an entire and connected piece, like a highly wrought story.

This author is remarkable also for his illustrations. He teaches by instances. He presents one example after another, each evident by itself, and each throwing some new light upon the subject, till the reader begins to anticipate for himself the truth to be inculcated.

Some opinion may be formed of the adaptation of this treatise to learners, from the circumstances under which it was composed. It was undertaken after the author became blind, and was dictated to a young man entirely without education, who by this means became an expert algebraist, and was able to render the author important services as an amanuensis. It was written originally in German. It has since been translated into Russian, French, and English, with notes and additions.

The entire work consists of two volumes octavo, and contains many things intended for the professed mathematician, rather than the general student. It was thought that a selection of such parts as would form an easy introduction to the science would be well received, and tend to promote a taste for analysis among the higher class of students, and to raise the character of mathematical learning.

Notwithstanding the high estimation in which this work has been held, it is scarcely to be met with in this country, and is very little known in England. On the continent of Europe this author is the constant theme of eulogy. His writings have the character of classics. They are regarded at the same time as the most profound and the most perspicuous, and as affording the finest models of analysis. They furnish the germs of the most approved elementary works on the different branches of this science. The constant reply of one of the first mathematicians* of France to those who consulted him upon the best method of studying mathematics was, "*study Euler.*" "It is needless," said he, "to accumulate books; true lovers of mathematics will always read Euler; because in his writings every thing is clear, distinct, and correct; because they swarm with excellent examples; and because it is always necessary to have recourse to the fountain head."

The selections here offered are from the first English edition. A few errors have been corrected and a few alterations made in the phraseology. In the original no questions were left to be performed by the learner. A collection was made by the English translator and subjoined at the end with references to the sections to which they relate. These have been mostly retained, and some new ones have been added.

Although this work is intended particularly for the algebraical student, it will be found to contain a clear and full explanation of the fundamental principles of arithmetic; vulgar frac-

* Lagrange.

tions, the doctrine of roots and powers, of the different kinds of proportion and progression, are treated in a manner, that can hardly fail to interest the learner and make him acquainted with the reason of those rules which he has so frequent occasion to apply.

A more extended work on Algebra formed after the same model is now in the press and will soon be published. This will be followed by other treatises upon the different branches of pure mathematics.

JOHN FARRAR,
Professor of Mathematics and Natural Philosophy in the University at Cambridge.

Cambridge, February, 1818.

CONTENTS.

SECTION I.

OF THE DIFFERENT METHODS OF CALCULATING SIMPLE QUANTITIES.

CHAPTER I.

Of Mathematics in general. 1

CHAPTER II.

Explanation of the Signs + Plus *and* — Minus. 3

CHAPTER III.

Of the Multiplication of Simple Quantities. 7

CHAPTER IV.

Of the nature of whole Numbers or Integers, with respect to their Factors. 11

CHAPTER V.

Of the Division of Simple Quantities. 13

CHAPTER VI.

Of the properties of Integers with respect to their Divisors. 16

CHAPTER VII.

Of Fractions in general. 19

CHAPTER VIII.

Of the properties of Fractions. 24

CHAPTER IX.

Of the Addition and Subtraction of Fractions. 26

CHAPTER X.

Of the Multiplication and Division of Fractions. 29

CHAPTER XI.

Of Square Numbers. 33

CHAPTER XII.

Of Square Roots, and of Irrational Numbers resulting from them. 35

CHAPTER XIII.

Of Impossible or Imaginary Quantities, which arise from the same source. 40

CHAPTER XIV.

Of Cubic Numbers. 43

CHAPTER XV.

Of Cube Roots, and of Irrational Numbers resulting from them. 44

CHAPTER XVI.

Of Powers in general. 47

CHAPTER XVII.

Of the calculation of Powers. 51

CHAPTER XVIII.

Of Roots with relation to Powers in general. 53

CHAPTER XIX.

Of the Method of representing Irrational Numbers by Fractional Exponents. 60

CHAPTER XX.

Of the different Methods of Calculation, and of their mutual Connexion. 58

SECTION SECOND.

OF THE DIFFERENT METHODS OF CALCULATING COMPOUND QUANTITIES.

CHAPTER I.

Of the Addition of Compound Quantities. 61

CHAPTER II.

Of the Subtraction of Compound Quantities. 63

CHAPTER III.

Of the Multiplication of Compound Quantities. 64

CHAPTER IV.

Of the Division of Compound Quantities. 70

CHAPTER V.

Of the Resolution of Fractions into infinite Series. 74

CHAPTER VI.

Of the Squares of Compound Quantities. 83

CHAPTER VII.

Of the Extraction of Roots applied to Compound Quantities. 86

CHAPTER VIII.
Of the calculation of Irrational Quantities. 90
CHAPTER IX.
Of Cubes, and of the Extraction of Cube Roots. 94
CHAPTER X.
Of the higher Powers of Compound Quantities. 96
CHAPTER XI.
Of the Transposition of the Letters, on which the demonstration of the preceding Rule is founded. 102
CHAPTER XII.
Of the expression of Irrational Powers by Infinite Series. 106
CHAPTER XIII.
Of the resolution of Negative Powers. 109

SECTION THIRD.

OF RATIOS AND PROPORTIONS.

CHAPTER I.
Of Arithmetical Ratio, or of the difference between two Numbers. 113
CHAPTER II.
Of Arithmetical Proportion. 115
CHAPTER III.
Of Arithmetical Progressions. 118
CHAPTER IV.
Of the Summation of Arithmetical Progressions. 122
CHAPTER V.
Of Geometrical Ratio. 126
CHAPTER VI.
Of the greatest Common Divisor of two given numbers. 128
CHAPTER VII.
Of Geometrical Proportions. 132
CHAPTER VIII.
Observations on the Rules of Proportion and their utility. 135
CHAPTER IX.
Of Compound Relations. 140

CHAPTER X.

Of Geometrical Progressions. 146

CHAPTER XI.

Of Infinite Decimal Fractions. 152

SECTION FOURTH.

OF ALGEBRAIC EQUATIONS, AND OF THE RESOLUTION OF THOSE EQUATIONS.

CHAPTER I.

Of the Solution of Problems in general. 158

CHAPTER II.

Of the Resolution of Simple Equations, or Equations of the first degree. 161

CHAPTER III.

Of the Solution of Questions relating to the preceding chapter. 165

CHAPTER IV.

Of the Resolution of two or more Equations of the First Degree. 175

CHAPTER V.

Of the Resolution of Pure Quadratic Equations. 184

CHAPTER VI.

Of the Resolution of Mixed Equations of the Second Degree. 190

CHAPTER VII.

Of the Nature of Equations of the Second Degree. 198

Questions for Practice. 204

New demonstrations of the fundamental propositions of the 5th book of Euclid's Elements. 217

INTRODUCTION

TO THE

ELEMENTS OF ALGEBRA.

SECTION I.

OF THE DIFFERENT METHODS OF CALCULATION APPLIED TO SIMPLE QUANTITIES.

CHAPTER I.

Of Mathematics in general.

ARTICLE I.

WHATEVER is capable of increase or diminution, is called *magnitude* or *quantity.*

A sum of money, for instance, is a quantity, since we may increase it or diminish it. The same may be said with respect to any given weight, and other things of this nature.

2. From this definition, it is evident, that there must be so many different kinds of magnitude as to render it difficult even to enumerate them: and this is the origin of the different branches of mathematics, each being employed on a particular kind of magnitude. Mathematics, in general, is the *science of quantity;* or the science which investigates the means of measuring quantity.

3. Now we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and pointing out their mutual relation. If it were proposed, for example, to determine the quantity of a sum of money, we should take some known piece of money (as a dollar, a crown, a ducat, or some other coin,) and shew how many of

these pieces are contained in the given sum. In the same manner, if it were proposed to determine the quantity of a weight, we should take a certain known weight; for example, a pound, an ounce, &c., and then shew how many times one of these weights is contained in that which we are endeavouring to ascertain. If we wished to measure any length or extension, we should make use of some known length, as a foot for example.

4. So that the determination, or the measure of magnitude of all kinds, is reduced to this: fix at pleasure upon any one known magnitude of the same species with that which is to be determined, and consider it as the *measure or unit;* then, determine the proportion of the proposed magnitude to this known measure. This proportion is always expressed by numbers; so that a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit.

5. From this it appears, that all magnitudes may be expressed by numbers; and that the foundation of all the mathematical sciences must be laid in a complete treatise on the science of numbers; and in an accurate examination of the different possible methods of calculation.

This fundamental part of mathematics is called analysis, or algebra.

6. In algebra then we consider only numbers which represent quantities, without regarding the different kinds of quantity. These are the subjects of other branches of the mathematics.

7. Arithmetic treats of numbers in particular, and is the *science of numbers properly so called;* but this science extends only to certain methods of calculation which occur in common practice: algebra, on the contrary, comprehends in general all the cases which can exist in the doctrine and calculation of numbers.

CHAPTER II.

Explanation of the signs + plus *and* — minus.

8. When we have to add one given number to another, this is indicated by the sign + which is placed before the second number, and is read *plus*. Thus $5 + 3$ signifies that we must add 3 to the number 5, and every one knows that the result is 8; in the same manner $12 + 7$ make 19; $25 + 16$ make 41; the sum of $25 + 41$ is 66, &c.

9. We also make use of the same sign + or *plus*, to connect several numbers together; for example, $7 + 5 + 9$ signifies that to the number 7 we must add 5 and also 9, which make 21. The reader will therefore understand what is meant by

$$8 + 5 + 13 + 11 + 1 + 3 + 10;$$

viz. the sum of all these numbers, which is 51.

10. All this is evident; and we have only to mention, that, in algebra, in order to generalize numbers, we represent them by letters, as a, b, c, d, &c. Thus the expression $a + b$ signifies the sum of two numbers, which we express by a and b, and these numbers may be either very great or very small. In the same manner, $f + m + b + x$, signifies the sum of the numbers represented by these four letters.

If we know therefore the numbers that are represented by letters, we shall at all times be able to find by arithmetic, the sum or value of similar expressions.

11. When it is required, on the contrary, to subtract one given number from another, this operation is denoted by the sign —, which signifies *minus*, and is placed before the number to be subtracted: thus $8 - 5$ signifies that the number 5 is to be taken from the number 8; which being done, there remains 3. In like manner $12 - 7$ is the same as 5; and $20 - 14$ is the same as 6, &c.

12. Sometimes also we may have several numbers to be subtracted from a single one; as for instance, $50 - 1 - 3 - 5 - 7 - 9$. This signifies, first, take 1 from 50, there remains 49; take 3 from that remainder, there will remain 46; take away 5, 41 remains; take away 7, 34 remains; lastly, from that take 9, and there

remains 25; this last remainder is the value of the expression. But as the numbers 1, 3, 5, 7, 9, are all to be subtracted, it is the same thing if we subtract their sum, which is 25, at once from 50, and the remainder will be 25 as before.

13. It is also very easy to determine the value of similar expressions, in which both the signs + *plus* and — *minus* are found: for example;

$$12 - 3 - 5 + 2 - 1 \text{ is the same as } 5.$$

We have only to collect separately the sum of the numbers that have the sign + before them, and subtract from it the sum of those that have the sign —. The sum of 12 and 2 is 14; that of 3, 5 and 1, is 9; now 9 being taken from 14, there remains 5.

14. It will be perceived from these examples that *the order in which we write the numbers is quite indifferent and arbitrary, provided the proper sign of each be preserved.* We might with equal propriety have arranged the expression in the preceding article thus; $12 + 2 - 5 - 3 - 1$, or $2 - 1 - 3 - 5 + 12$, or $2 + 12 - 3 - 1 - 5$, or in still different orders. It must be observed, that in the expression proposed, the sign + is supposed to be placed before the number 12.

15. It will not be attended with any more difficulty, if, in order to generalize these operations, we make use of letters instead of real numbers. It is evident, for example, that

$$a - b - c + d - e$$

signifies that we have numbers expressed by a and d. and that from these numbers, or from their sum, we must subtract the numbers expressed by the letters b, c, e, which have before them the sign —.

16. Hence it is absolutely necessary to consider what sign is prefixed to each number: for *in algebra, simple quantities are numbers considered with regard to the signs which precede, or affect them.* Further, we call those *positive quantities,* before which the sign + is found; and those are called *negative quantities,* which are affected with the sign —.

17. The manner in which we generally calculate a person's property, is a good illustration of what has just been said. We denote what a man really possesses by positive numbers, using, or understaning the sign +; whereas his debts are represent-

ed by negative numbers, or by using the sign —. Thus, when it is said of any one that he has 100 crowns, but owes 50, this means that his property really amounts to 100 — 50 ; or, which is the same thing, + 100 — 50, that is to say 50.

18. As negative numbers may be considered as debts, because positive numbers represent real possessions, we may say that negative numbers are less than nothing. Thus, when a man has nothing in the world, and even owes 50 crowns, it is certain that he has 50 crowns less than nothing ; for if any one were to make him a present of 50 crowns to pay his debts, he would still be only at the point nothing, though really richer than before.

19. In the same manner therefore as positive numbers are incontestably greater than nothing, negative numbers are less than nothing.* Now we obtain positive numbers by adding 1 to 0, that is to say, to nothing; and by continuing always to increase thus from unity. This is the origin of the series of numbers called *natural numbers ;* the following are the leading terms of this series :

$$0, +1, +2, +3, +4, +5, +6, +7, +8, +9, +10,$$

and so on to infinity.

But if instead of continuing this series by successive additions, we continued it in the opposite direction, by perpetually subtracting unity, we should have the series of negative numbers :

$$0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10,$$

and so on to infinity.

* By being less than nothing is meant simply that they are of such a nature as to cancel or destroy an equal number with the sign plus before it, so that — 4, or — a is as really a positive thing, and is as easily conceived, as + 4 or + a. The quantity 4 or a may be considered independently of its sign. The sign + implies that this quantity is to be added, and the sign — that it is to be subtracted. This subject may be illustrated by the scale of a thermometer. After observing the mercury to stand at 50°, for instance, I am told, that it has changed 4°, I have a distinct idea of the portion of the scale denoted by four of its divisions, without applying them in any particular direction. But when I am further informed that this change of the thermometer is — or *subtractive* with respect to its former state, I then understand that the mercury stands at 46°, whereas it would be at 54° if the change had been + or additive.

20. All these numbers, whether positive or negative, have the known appellation of whole numbers, or *integers,* which consequently are either greater or less than nothing. We call them *integers,* to distinguish them from fractions, and from several other kinds of numbers of which we shall hereafter speak. For instance, 50 being greater by an entire unit than 49, it is easy to comprehend that there may be between 49 and 50 an infinity of intermediate numbers, all greater than 49, and yet all less than 50. We need only imagine two lines, one 50 feet the other 49 feet long, and it is evident that there may be drawn an infinite number of lines all longer than 49 feet, and yet shorter than 50.

21. It is of the utmost importance through the whole of algebra, that a precise idea be formed of those negative quantities about which we have been speaking. I shall content myself with remarking here that all such expressions, as

$$+1-1, +2-2, +3-3, +4-4, \&c.$$

are equal to 0 or nothing. And that

$$+2-5 \text{ is equal to } -3.$$

For if a person has 2 crowns, and owes 5, he has not only nothing, but still owes 3 crowns: in the same manner,

$$7-12 \text{ is equal to } -5. \text{ and } 25-40 \text{ is equal to } -15.$$

22. The same observations hold true, when, to make the expression more general, letters are used instead of numbers: 0 or nothing will always be the value of $+a-a$. If we wish to know the value $+a-b$ two cases are to be considered.

The first is when a is greater than b; b must then be subtracted from a, and the remainder (before which is placed or understood to be placed the sign $+$) shews the value sought.

The second case is that in which a is less than b: here a is to be subtracted from b, and the remainder being made negative, by placing before it the sign $-$, will be the value sought.

CHAPTER III.

Of the Multiplication of Simple Quantities.

23. When there are two or more equal numbers to be added together, the expression of their sum may be abridged; for example,

$a+a$ is the same with $2\times a$,
$a+a+a$ $3\times a$,
$a+a+a+a$ $4\times a$, and so on; where $\times$ is the sign of multiplication. In this manner we may form an idea of multiplication; and it is to be observed that,

$2\times a$ signifies 2 times, or twice a
$3\times a$ 3 times, or thrice a
$4\times a$ 4 times a, &c.

24. *If therefore a number expressed by a letter is to be multiplied by any other number, we simply put that number before the letter;* thus,

a multiplied by 20 is expressed by $20a$, and
b multiplied by 30 gives $30b$, &c.

It is evident also that c taken once, or $1c$, is just c.

25. Further, it is extremely easy to multiply such products again by other numbers; for example:

2 times, or twice $3a$ makes $6a$,
3 times, or thrice $4b$ makes $12b$,
5 times $7x$ makes $35x$,

and these products may be still multiplied by other numbers at pleasure.

26. *When the number, by which we are to multiply, is also represented by a letter, we place it immediately before the other letter;* thus, in multiplying b by a, the product is written ab; and pq will be the product of the multiplication of the number q by p. If we multiply this pq again by a, we shall obtain apq.

27. It may be remarked here, that *the order in which the letters are joined together is indifferent;* that ab is the same thing as ba; for b multiplied by a produces as much as a multiplied by b. To understand this, we have only to substitute for a and b

known numbers, as 3 and 4 ; and the truth will be self-evident ; for 3 times 4 is the same as 4 times 3.

28. It will not be difficult to perceive, that when you have to put numbers, in the place of letters joined together, as we have described, they cannot be written in the same manner by putting them one after the other. For if we were to write 34 for 3 times 4, we should have 34 and not 12. When therefore it is required to multiply common numbers, we must separate them by the sign $\times$, or points : thus, 3×4, or $3 \cdot 4$, signifies 3 times 4, that is 12. So, 1×2 is equal to 2 ; and $1 \times 2 \times 3$ makes 6. In like manner $1 \times 2 \times 3 \times 4 \times 56$ makes 1344 ; and $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$ is equal to 3628800, &c.

29. In the same manner, we may discover the value of an expression of this form, $5 \cdot 7 \cdot 8\, a\, b\, c\, d$. It shews that 5 must be multiplied by 7, and that this product is to be again multiplied by 8 ; that we are then to multiply this product of the three numbers by a, next by b, and then by c, and lastly by d. It may be observed also, that instead of $5 \times 7 \times 8$ we may write its value, 280 ; for we obtain this number when we multiply the product of 5 by 7 or 35, by 8.

30. The results which arise from the multiplication of two or more numbers are called *products ;* and the numbers, or individual letters, are called *factors*.

31. Hitherto we have considered only positive numbers, and there can be no doubt, but that the products which we have seen arise are positive also : viz. $+ a$ by $+ b$ must necessarily give $+ a b$. But we must separately examine what the multiplication of $+ a$ by $- b$, and of $- a$ by $- b$, will produce.

32. Let us begin by multiplying $- a$ by 3 or $+ 3$; now since $- a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $- 3 a$. So if we multiply $- a$ by $+ b$, we shall obtain $- b a$, or, which is the same thing, $- a b$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative ; and lay it down as a rule, that $+$ by $+$ makes $+$, or *plus*, and that on the contrary $+$ by $-$, or $-$ by $+$ gives $-$, or *minus*.

33. It remains to resolve the case in which — is multiplied by —; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign +, or the sign —, is to be placed before the product; all we know is, that it must be one or the other of these signs. Now I say that it cannot be the sign —: for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently we have the following rule: — multiplied by — produces +, in the same manner as + multiplied by +.*

* It is a subject of great embarrassment and perplexity to learners to conceive how the product of two negative quantities should be positive. This arises from the idea they receive of the nature of multiplication as explained and applied in arithmetic, where positive quantities only are employed. The term is used in a more enlarged sense when negative quantities are concerned, as may be shown without making use of letters. If I wished to multiply, for instance, $9-5$ (or 9 diminished by 5) by 3, I should first find the product of 9 by 3 or 27. But this is evidently taking the multiplicand too great by 5, and of course the product too great by 3 times 5; I accordingly write for the product $27-15$, equivalent to 12, which is the product that would arise from first performing the subtraction indicated by the sign —, and using the result as the multiplicand. Thus,

Multiplicand	$9-5$	which is equal to	4
Multiplier	3		3
Product	$27-15$	which is equal to	12

Let us now take for the multiplier the quantity $7-4$, which is equivalent to 3. We multiply, in the first place, by 7, in the manner that we have just done by 3, and the result is $63-35$. But as the multiplier is 7 diminished by 4, multiplying by 7 must give 4 times too much. Accordingly we take 4 times the multiplicand, or $36-20$, and subtract this from $63-35$, or 7 times the multiplicand. Now in making this subtraction it is to be observed that the subtrahend $36-20$ is 36 diminished by 20, and if we subtract 36 we take away too much by 20, and must therefore add this latter quantity. Consequently the true product will be $63-35-36+20$, equivalent to 12, as before. Thus this mode of proceeding gives the same result as that obtained by first performing the subtractions indicated in the latter term of the multiplicand and multiplier. The several steps in each case are as follows:

Multiplicand	$9-5$	which is equal to	4
Multiplier	$7-4$	which is equal to	3
	$63-35$	Product	12
	$-36+20$		
Product	$63-35-36+20$ or $83-71$, that is 12.		

34. The rules which we have explained are expressed more briefly as follows:

Like signs, multiplied together, give +; unlike or contrary signs

Thus we see that 7 or $+7$ by -5 gives -35, and -4 by $+9$ gives -36, and -4 by -5 gives $+20$. The same general reasoning will apply when letters are used instead of numbers.

$$\begin{array}{ll} \text{Multiplicand} & a-b \\ \text{Multiplier} & \underline{c-d} \\ & ac-bc \\ & \underline{\quad -ad+bd} \\ \text{Product} & ac-bc-ad+bd. \end{array}$$

We say in this case that, when we multiply a by c we take the multiplicand too great by b, and must therefore diminish the result $a\,c$ by the product of b by c or $b\,c$. So also in multiplying the two terms of the multiplicand by c, we have taken the multiplier too great by d, and must therefore diminish the result $a\,c-b\,c$ by the product of $a-b$ by d, or $a\,d-b\,d$. But if we subtract the whole of $a\,d$, we subtract too much by $b\,d$; $b\,d$ must accordingly be added.

The rule for negative quantities here illustrated is not necessary where mere numbers are employed, because the subtraction indicated may always be performed. But this cannot be done with respect to letters which stand for no particular values, but are intended as general expressions of quantities.

The truth of the rule may be shown also when applied to quantities taken singly. We say that multiplying one quantity by another is taking one as many times as there are units in the other, and the result is the same, whichever of the quantities be taken for the multiplicand. Thus multiplying 9 by 3 is taking 9 three times, or, which is the same thing, taking 3 nine times (Arith. 27). But in arithmetic, quantities are always taken affirmatively, that is additively. When therefore we take 9 or $+9$ three times additively, or $+3$ nine times additively, the result will evidently be additive or $+27$. When on the contrary one of the factors is negative, as for instance, in multiplying -5 by $+3$; in this case, -5 is to be taken three times additively, and -5 added to -5 added to -5 is clearly -15. So also if we consider $+3$ as the multiplicand, then $+3$ is to be taken five times subtractively; now 3 taken subtractively once (or which is the same thing 3×-1) is equivalent to -3, taken subtractively twice is -6, three times is -9, five times is -15. But, when the multiplicand and multiplier are both negative, as in the case of multiplying -5 by -4; here a subtractive quantity is to be taken subtractively, that is, we are to take away successively a diminishing or lessening quantity, which is certainly equivalent to adding an increasing quantity. Thus if we take away -5 once, we augment the sum with which it is to be connected by $+5$; if we take away -5 twice, we make the augmentation $+10$; if four times, $+20$; that is, -5×-4 is equivalent to $+20$.

give —. Thus, when it is required to multiply the following numbers; $+a$, $-b$, $-c$, $+d$; we have first $+a$ multiplied by $-b$, which makes $-ab$; this by $-c$, gives $+abc$; and this by $+d$, gives $+abcd$.

35. The difficulties with respect to the signs being removed, we have only to shew how to multiply numbers that are themselves products. If we were, for instance, to multiply the number ab by the number cd, the product would be $abcd$, and it is obtained by multiplying first ab by c, and then the result of that multiplication by d. Or, if we had to multiply 36 by 12; since 12 is equal to 3 times 4, we should only multiply 36 first by 3, and then the product 108 by 4, in order to have the whole product of the multiplication of 12 by 36, which is consequently 432.

36. But if we wished to multiply $5ab$ by $3cd$, we might write $3cd \times 5ab$; however, as in the present instance the order of the numbers to be multiplied is indifferent, it will be better, as is also the custom, to place the common numbers before the letters, and to express the product thus: $5 \times 3abcd$, or $15abcd$; since 5 times 3 is 15.

So if we had to multiply $12pqr$ by $7xy$, we should obtain $12 \times 7pqrxy$, or $84pqrxy$.

CHAPTER IV.

Of the nature of whole numbers or integers, with respect to their factors.

37. We have observed that a product is generated by the multiplication of two or more numbers together, and that these numbers are called *factors*. Thus the numbers a, b, c, d, are the factors of the product $abcd$.

38. If, therefore, we consider all whole numbers as products of two or more numbers multiplied together, we shall soon find that some cannot result from such a multiplication, and consequently have not any factors; while others may be the products of two or more multiplied together, and may consequently have two or more factors. Thus, 4 is produced by 2×2; 6 by 2×3; 8 by $2 \times 2 \times 2$; or 27 by $3 \times 3 \times 3$; and 10 by 2×5, &c.

39. But, on the other hand, the numbers, 2, 3, 5, 7, 11, 13, 17, &c., cannot be represented in the same manner by factors, unless for that purpose we make use of unity, and represent 2, for instance, by 1×2. Now the numbers which are multiplied by 1, remaining the same, it is not proper to reckon unity as a factor.

All numbers therefore, such as 2, 3, 5, 7, 11, 13, 17, &c. which cannot be represented by factors, are called *simple,* or *prime numbers;* whereas others, as 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, &c. which may be represented by factors, are called *compound numbers.*

40. *Simple* or *prime numbers* deserve therefore particular attention, since they do not result from the multiplication of two or more numbers. It is particularly worthy of observation that if we write these numbers in succession as they follow each other thus;

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, &c.

we can trace no regular order; their increments are sometimes greater, sometimes less; and hitherto no one has been able to discover whether they follow any certain law or not.

41. *All compound numbers, which may be represented by factors, result from the prime numbers above mentioned; that is to say, all their factors are prime numbers.* For, if we find a factor which is not a prime number, it may always be decomposed and represented by two or more prime numbers. When we have represented, for instance, the number 30 by 5×6, it is evident that 6 not being a prime number, but being produced by 2×3, we might have represented 30 by $5 \times 2 \times 3$, or by $2 \times 3 \times 5$; that is to say, by factors, which are all prime numbers.

42. If we now consider those compound numbers which may be resolved into prime numbers, we shall observe a great difference among them; we shall find that some have only two factors, that others have three, and others a still greater number. We have already seen, for example, that

4	is the same as 2×2,	6	is the same as 2×3,
8	$2\times2\times2$,	9	3×3,
10	2×5,	12	$2\times3\times2$,
14	2×7,	15	$3+5$,
16	$2\times2\times2\times2$,	and so on.	

43. Hence it is easy to find a method for analysing any number, or resolving it into its simple factors. Let there be proposed, for instance, the number 360; we shall represent it first by 2×180. Now 180 is equal to 2×90, and

$$\left.\begin{array}{r}90\\45\\15\end{array}\right\}\text{ is the same as }\left\{\begin{array}{l}2\times 45,\\3\times 15,\text{ and lastly}\\3\times\ \ 5.\end{array}\right.$$

So that the number 360 may be represented by these simple factors, $2 \times 2 \times 2 \times 3 \times 3 \times 5$; since all these numbers multiplied together produce 360.

44. This shews, that the prime numbers cannot be divided by other numbers, and on the other hand, that *the simple factors of compound numbers are found, most conveniently and with the greatest certainty, by seeking the simple, or prime numbers, by which those compound numbers are divisible.* But for this, *division* is necessary; we shall therefore explain the rules of that operation in the following chapter.

CHAPTER V.

Of the Division of Simple Quantities.

45. When a number is to be separated into two, three, or more equal parts, it is done by means of *division*, which enables us to determine the magnitude of one of those parts. When we wish, for example, to separate the number 12 into three equal parts, we find by division that each of those parts is equal to 4.

The following terms are made use of in this operation. The number, which is to be decompounded or divided, is called the *dividend*; the number of equal parts sought is called the *divisor*; the magnitude of one of those parts, determined by the division, is called the *quotient*; thus, in the above example;

12 is the dividend,
3 is the divisor, and
4 is the quotient.

46. It follows from this, that if we divide a number by 2, or into two equal parts, one of those parts, or the quotient, taken twice, makes exactly the number proposed; and, in the same

manner, if we have a number to be divided by 3, the quotient taken thrice must give the same number again. In general, *the multiplication of the quotient by the divisor must always reproduce the dividend.*

47. It is for this reason that division is called a rule, which teaches us to find a number or quotient, which, being multiplied by the divisor, will exactly produce the dividend. For example, if 35 is to be divided by 5, we seek a number which, multiplied by 5, will produce 35. Now this number is 7, since 5 times 7 is 35. The mode of expression, employed in this reasoning, is; 5 in 35, 7 times; and 5 times 7 makes 35.

48. The dividend therefore may be considered as a product, of which one of the factors is the divisor, and the other the quotient. Thus, supposing we have 63 to divide by 7, we endeavour to find such a product, that taking 7 for one of its factors, the other factor multiplied by this may exactly give 63. Now 7×9 is such a product, and consequently 9 is the quotient obtained when we divide 63 by 7.

49. In general, if we have to divide a number $a\,b$ by a, it is evident that the quotient will be b; for a multiplied by b gives the dividend $a\,b$. It is clear also, that if we had to divide $a\,b$ by b, the quotient would be a. And in all examples of division that can be proposed, if we divide the dividend by the quotient, we shall again obtain the divisor; for as 24 divided by 4 gives 6, so 24 divided by 6 will give 4.

50. As *the whole operation consists in representing the dividend by two factors, of which one shall be equal to the divisor, the other to the quotient;* the following examples will be easily understood. I say first, that the dividend $a\,b\,c$, divided by a, gives $b\,c$; for a, multiplied by $b\,c$, produces $a\,b\,c$: in the same manner $a\,b\,c$, being divided by b, we shall have $a\,c$; and $a\,b\,c$, divided by $a\,c$, gives b. I say also, that $12\,m\,n$, divided by $3\,m$, gives $4\,n$; for $3\,m$, multiplied by $4\,n$, makes $12\,m\,n$. But if this same number $12\,m\,n$ had been divided by 12, we should have obtained the quotient $m\,n$.

51. Since every number a may be expressed by $1\,a$ or *one* a, it is evident that if we had to divide a or $1\,a$ by 1, the quotient would

be the same number a. But, on the contrary, if the same number a, or $1\,a$ is to be divided by a, the quotient will be 1.

52. It often happens that we cannot represent the dividend as the product of two factors, of which one is equal to the divisor; and then the division cannot be performed in the manner we have described.

When we have, for example, 24 to be divided by 7, it is at first sight obvious, that the number 7 is not a factor of 24; for the product of 7×3 is only 21, and consequently too small, and 7×4 makes 28, which is greater than 24. We discover however from this, that the quotient must be greater than 3, and less than 4. In order therefore to determine it exactly, we employ another species of numbers, which are called *fractions*, and which we shall consider in one of the following chapters.

53. Until the use of fractions is considered, it is usual to rest satisfied with the whole number which approaches nearest to the true quotient, but at the same time paying attention to the *remainder* which is left; thus we say, 7 in 24, 3 times, and the remainder is 3, because 3 times 7 produces only 21, which is 3 less than 24. We may consider the following examples in the same manner:

$$\begin{array}{r} 6)34(5, \\ 30 \\ \hline 4 \end{array}$$

that is to say, the divisor is 6, the dividend 34, the quotient 5, and the remainder 4.

$$\begin{array}{r} 9)41(4, \\ 36 \\ \hline 5 \end{array}$$

here the divisor is 9, the dividend 41, the quotient 4, and the remainder 5.

The following rule is to be observed in examples where there is a remainder.

54. *If we multiply the divisor by the quotient, and to the product add the remainder, we must obtain the dividend;* this is the method of proving division, and of discovering whether the calculation is right or not. Thus, in the first of the two last examples, if we multiply 6 by 5, and to the product 30 add the remainder 4, we obtain 34, or the dividend. And in the last example, if we multiply the divisor 9 by the quotient 4, and to the product 36 add the remainder 5, we obtain the dividend 41.

55. Lastly, it is necessary to remark here, with regard to the signs + *plus* and — *minus*, that if we divide $+ab$ by $+a$, the quotient will be $+b$, which is evident. But if we divide $+ab$ by $-a$, the quotient will be $-b$; because $-a \times -b$ gives $+ab$. If the dividend is $-ab$, and is to be divided by the divisor $+a$, the quotient will be $-b$; because it is $-b$, which, multiplied by $+a$, makes $-ab$. Lastly, if we have to divide the dividend $-ab$ by the divisor $-a$, the quotient will be $+b$; for the dividend $-ab$ is the product of $-a$ by $+b$.

56. *With regard therefore to the signs + and —, division admits the same rules that we have seen applied in multiplication*; viz.

+ by + requires +; + by — requires —;

— by + requires —; — by — requires +:

or in a few words, *like signs give plus, unlike signs give minus.*

57. Thus, when we divide $18pq$ by $-3p$, the quotient is $-6q$. Further;

$-30xy$, divided by $+6y$, gives $-5x$, and

$-54abc$, divided by $-9b$, gives $+6ac$;

for in this last example, $-9b$, multiplied by $+6ac$, makes $-6 \times 9abc$, or $-54abc$. But we have said enough on the division of simple quantities; we shall therefore hasten to the explanation of fractions, after having added some farther remarks on the nature of numbers, with respect to their divisors.

CHAPTER VI.

Of the properties of integers with respect to their divisors.

58. As we have seen that some numbers are divisible by certain divisors, while others are not; in order that we may obtain a more particular knowledge of numbers, this difference must be carefully observed, both by distinguishing the numbers that are divisible by divisors from those which are not, and by considering the remainder that is left in the division of the latter. For this purpose let us examine the divisors;

2, 3, 4, 5, 6, 7, 8, 9, 10, &c.

59. First, let the divisor be 2; the numbers divisible by it are 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, &c. which, it appears

increase always by two. These numbers, as far as they can be continued, are called *even numbers.* But there are other numbers, namely,

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, &c.,

which are uniformly less or greater than the former by unity, and which cannot be divided by 2, without the remainder 1; these are called *odd numbers.*

The even numbers are all comprehended in the general expression $2a$; for they are all obtained by successively substituting for a the integers 1, 2, 3, 4, 5, 6, 7, &c., and hence it follows that the odd numbers are all comprehended in the expression $2a+1$, because $2a+1$ is greater by unity than the even number $2a$.

60. In the second place, let the number 3 be the divisor; the numbers divisible by it are,

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, and so on; and these numbers may be represented by the expression $3a$; for $3a$ divided by 3 gives the quotient a without a remainder. All other numbers, which we would divide by 3, will give 1 or 2 for a remainder, and are consequently of two kinds. Those which, after the division leave the remainder 1, are;

1, 4, 7, 10, 13, 16, 19, &c.,

and are contained in the expression $3a+1$; but the other kind, where the numbers give the remainder 2, are;

2, 5, 8, 11, 14, 17, 20, &c.,

and they may be generally expressed by $3a+2$: so that all numbers may be expressed either by $3a$, or by $3a+1$, or by $3a+2$.

61. Let us now suppose that 4 is the divisor under consideration: the numbers which it divides are;

4, 8, 12, 16, 20, 24, &c.,

which increase uniformly by 4, and are comprehended in the expression $4a$. All other numbers, that is, those which are not divisible by 4, may leave the remainder 1, or be greater than the former by 1: as

1, 5, 9, 13, 17, 21, 25, &c.,

and consequently may be comprehended in the expression $4a+1$: or they may give the remainder 2; as

2, 6, 10, 14, 18, 22, 26, &c.,

and be expressed by $4a+2$; or, lastly, they may give the remainder 3; as

$$3, 7, 11, 15, 19, 23, 27, \&c.,$$

and may be represented by the expression $4a+3$.

All possible integral numbers are therefore contained in one or other of these four expressions;

$$4a,\ 4a+1,\ 4a+2,\ 4a+3.$$

62. It is nearly the same when the divisor is 5; for all numbers which can be divided by it are comprehended in the expression $5a$, and those which cannot be divided by 5, are reducible to one of the following expressions:

$$5a+1,\ 5a+2,\ 5a+3,\ 5a+4;$$

and we may go on in the same manner and consider the greatest divisors.

63. It is proper to recollect here what has been already said on the resolution of numbers into their simple factors; for every number, among the factors of which is found,

2, or 3, or 4, or 5, or 7,

or any other number, will be divisible by those numbers. For example; 60 being equal to $2 \times 2 \times 3 \times 5$, it is evident that 60 is divisible by 2, and by 3, and by 5.

64. Further, as the general expression $abcd$ is not only divisible by a, and b, and c, and d, but also by

ab, ac, ad, bc, bd, cd, and by
abc, abd, acd, bcd, and lastly by
$abcd$, that is to say, its own value;

it follows that 60, or $2 \times 2 \times 3 \times 5$, may be divided not only by these simple numbers, but also by those which are composed of two of them; that is to say, by 4, 6, 10, 15: and also by those which are composed of three of the simple factors, that is to say, by 12, 20, 30, and lastly by 60 itself.

65. *When, therefore, we have represented any number, assumed at pleasure, by its simple factors, it will be very easy to shew all the numbers by which it is divisible. For we have only, first, to take the simple factors one by one, and then to multiply them together two by two, three by three, four by four, &c., till we arrive at the number proposed.*

66. It must here be particularly observed; that every number is divisible by 1; and also that every number is divisible by

itself; so that every number has at least two factors, or divisors, the number itself and unity: but every number, which has no other divisor than these two, belongs to the class of numbers, which we have before called *simple*, or *prime numbers*.

All numbers, except these, have, beside unity and themselves, other divisors, as may be seen from the following table, in which are placed under each number all its divisors.

TABLE.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	3	2	5	2	7	2	3	2	11	2	13	2	3	2	17	2	19	2
			4		3		4	9	5		3		7	5	4		3		4
					6		8		10		4		14	15	8		6		5
											6				16		9		10
											12						18		20
1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6
P.	P.	P.		P.		P.				P.		P.				P.		P.	

67. Lastly, it ought to be observed that 0, or *nothing*, may be considered as a number which has the property of being divisible by all possible numbers; because by whatever number a we divide 0, the quotient is always 0; for it must be remarked that the multiplication of any number by *nothing* produces nothing, and therefore 0 times a, or $0\,a$, is 0.

CHAPTER VII.

Of Fractions in general.

68. When a number, as 7 for instance, is said not to be divisible by another number, let us suppose by 3, this only means, that the quotient cannot be expressed by an integral number: and it must not be thought by any means that it is

impossible to form an idea of that quotient. Only imagine a line of 7 feet in length, no one can doubt the possibility of dividing this line into 3 equal parts, and of forming a notion of the length of one of those parts.

69. Since therefore we may form a precise idea of the quotient obtained in similar cases, though that quotient is not an integral number, this leads us to consider a particular species of numbers, called *fractions,* or *broken numbers.* The instance adduced furnishes an illustration. If we have to divide 7 by 3, we easily conceive the quotient which should result, and express it by $\frac{7}{3}$; placing the divisor under the dividend, and separating the two numbers by a stroke, or line.

70. So, *in general, when the number* a *is to be divided by the number* b, *we represent the quotient by* $\frac{a}{b}$, *and call this form of expression a fraction.* We cannot therefore give a better idea of a fraction $\frac{a}{b}$, than by saying that we thus express the quotient resulting from the division of the upper number by the lower. We must remember also, that in all fractions the lower number is called the *denominator,* and that above the line the *numerator.*

71. In the above fraction, $\frac{7}{3}$, which we read *seven thirds,* 7 is the numerator, and 3 the denominator. We must also read $\frac{2}{3}$, two thirds; $\frac{3}{4}$, three fourths; $\frac{3}{8}$, three eighths; $\frac{12}{100}$, twelve hundredths; and $\frac{1}{2}$, one half.

72. In order to obtain a more perfect knowledge of the nature of fractions, we shall begin by considering the case in which the numerator is equal to the denominator, as in $\frac{a}{a}$. Now, since this expresses the quotient obtained by dividing a by a, it is evident that this quotient is exactly unity, and that consequently this fraction $\frac{a}{a}$ is equal to 1, or one integer; for the same reason, all the following fractions,

$$\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}, \text{\&c.},$$

are equal to one another, each being equal to 1, or one integer.

73. We have seen that a fraction, whose numerator is equal to the denominator, is equal to unity. All fractions therefore, whose numerators are less than the denominators, have a value

less than unity. For, if I have a number to be divided by another which is greater, the result must necessarily be less than 1; if we cut a line, for example, two feet long, into three parts, one of those parts will unquestionably be shorter than a foot: it is evident then, that $\frac{2}{3}$ is less than 1, for the same reason, that the numerator 2 is less than the denominator 3.

74. If the numerator, on the contrary, be greater than the denominator, the value of the fraction is greater than unity. Thus $\frac{3}{2}$ is greater than 1, for $\frac{3}{2}$ is equal to $\frac{2}{2}$ together with $\frac{1}{2}$. Now $\frac{2}{2}$ is exactly 1, consequently $\frac{3}{2}$ is equal to $1+\frac{1}{2}$, that is, to an integer and a half. In the same manner $\frac{4}{3}$ is equal to $1\frac{1}{3}$, $\frac{5}{3}$ to $1\frac{2}{3}$, and $\frac{7}{3}$ to $2\frac{1}{3}$. And in general, it is sufficient in such cases to divide the upper number by the lower, and to add to the quotient a fraction having the remainder for the numerator, and the divisor for the denominator. If the given fraction were, for example, $\frac{43}{12}$, we should have for the quotient 3, and 7 for the remainder; whence we conclude that $\frac{43}{12}$ is the same as $3\frac{7}{12}$.

75. Thus we see how fractions, whose numerators are greater than the denominators, are resolved into two parts; one of which is an integer, and the other a fractional number, having the numerator less than the denominator. Such fractions as contain one or more integers, are called *improper fractions*, to distinguish them from fractions properly so called, which, having the numerator less than the denominator, are less than unity, or than an integer.

76. The nature of fractions is frequently considered in another way, which may throw additional light on the subject. If we consider, for example, the fraction $\frac{3}{4}$, it is evident that it is three times greater than $\frac{1}{4}$. Now this fraction $\frac{1}{4}$ means, that if we divide 1 into 4 equal parts, this will be the value of one of those parts; it is obvious then, that by taking 3 of those parts, we shall have the value of the fraction $\frac{3}{4}$.

In the same manner we may consider every other fraction; for example, $\frac{7}{12}$; if we divide unity into 12 equal parts, 7 of those parts will be equal to this fraction.

77. From this manner of considering fractions, the expressions *numerator* and *denominator* are derived. For, as in the

preceding fraction $\frac{7}{12}$, the number under the line shews that 12 is the number of parts into which unity is to be divided; and as it may be said to denote, or name the parts, it has not improperly been called the *denominator.*

Further, as the upper number, namely 7, shews that, in order have the value of the fraction, we must take, or collect 7 of those parts, and therefore may be said to reckon, or number them, it has been thought proper to call the number above the line the *numerator.*

78. As it is easy to understand what $\frac{3}{4}$ is, when we know the signification of $\frac{1}{4}$, we may consider the fractions, whose numerator is unity, as the foundation of all others. Such are the fractions,

$$\tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \tfrac{1}{7},\ \tfrac{1}{8},\ \tfrac{1}{9},\ \tfrac{1}{10},\ \tfrac{1}{11},\ \tfrac{1}{12},\ \&c.,$$

and it is observable that these fractions go on continually diminishing; for the more you divide an integer, or the greater the number of parts into which you distribute it, the less does each of those parts become. Thus $\frac{1}{100}$ is less than $\frac{1}{10}$; $\frac{1}{1000}$ is less than $\frac{1}{100}$; and $\frac{1}{10000}$ is less than $\frac{1}{1000}$.

79. As we have seen, that the more we increase the denominator of such fractions, the less their values become; it may be asked, whether it is not possible to make the denominator so great, that the fraction shall be reduced to nothing? I answer, no; for into whatever number of parts unity (the length of a foot for instance) is divided; let those parts be ever so small, they will still preserve a certain magnitude, and therefore can never be absolutely reduced to nothing.

80. It is true, if we divide the length of a foot into 1000 parts; those parts will not easily fall under the cognizance of our senses: but view them through a good microscope, and each of them will appear large enough to be subdivided into 100 parts, and more.

At present, however, we have nothing to do with what depends on ourselves, or with what we are capable of performing, and what our eyes can perceive; the question is rather, what is possible in itself. And, in this sense of the word, it is certain, that however great we suppose the denominator, the fraction will never entirely vanish, or become equal to 0.

81. We never therefore arrive completely at nothing, however great the denominator may be; and these fractions always preserving a certain value, we may continue the series of fractions in the 78th article without interruption. This circumstance has introduced the expression, that the denominator must be *infinite*, or infinitely great, in order that the fraction may be reduced to 0, or to nothing; and the word *infinite* in reality signifies here, that we should never arrive at the end of the series of the above mentioned *fractions*.

82. To express this idea, which is extremely well founded, we make use of the sign ∞, which consequently indicates a number infinitely great; and we may therefore say that this fraction $\frac{1}{\infty}$ is really nothing, for the very reason that a fraction cannot be reduced to nothing, until the denominator has been increased to *infinity*.

83. It is the more necessary to pay attention to this idea of infinity, as it is derived from the first foundations of our knowledge, and as it will be of the greatest importance in the following part of this treatise.

We may here deduce from it a few consequences, that are extremely curious and worthy of attention. The fraction $\frac{1}{\infty}$ represents the quotient resulting from the division of the dividend 1 by the divisor ∞. Now we know that if we divide the dividend 1 by the quotient $\frac{1}{\infty}$, which is equal to 0, we obtain again the divisor ∞: hence we acquire a new idea of infinity; we learn that it arises from the division of 1 by 0; and we are therefore entitled to say, that 1 divided by 0 expresses a number infinitely great, or ∞.

84. It may be necessary also in this place to correct the mistake of those who assert, that a number infinitely great is not susceptible of increase. This opinion is inconsistent with the just principles which we have laid down; for $\frac{1}{0}$ signifying a number infinitely great, and $\frac{2}{0}$ being incontestably the double of $\frac{1}{0}$, it is evident that a number, though infinitely great, may still become two or more times greater.

CHAPTER VIII.

Of the properties of Fractions.

85. We have already seen, that each of the fractions,

$$\tfrac{2}{2}, \tfrac{3}{3}, \tfrac{4}{4}, \tfrac{5}{5}, \tfrac{6}{6}, \tfrac{7}{7}, \tfrac{8}{8}, \text{\&c.},$$

makes an integer, and that consequently they are all equal to one another. The same equality exists in the following fractions,

$$\tfrac{2}{1}, \tfrac{4}{2}, \tfrac{6}{3}, \tfrac{8}{4}, \tfrac{10}{5}, \tfrac{12}{6}, \text{\&c.},$$

each of them making two integers; for the numerator of each, divided by its denominator, gives 2. So all the fractions

$$\tfrac{3}{1}, \tfrac{6}{2}, \tfrac{9}{3}, \tfrac{12}{4}, \tfrac{15}{5}, \tfrac{18}{6}, \text{\&c.},$$

are equal to one another, since 3 is their common value.

86. We may likewise represent the value of any fraction, in an infinite variety of ways. For *if we multiply both the numerator and the denominator of a fraction by the same number, which may be assumed at pleasure, this fraction will still preserve the same value.* For this reason all the fractions

$$\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \tfrac{5}{10}, \tfrac{6}{12}, \tfrac{7}{14}, \tfrac{8}{16}, \tfrac{9}{18}, \tfrac{10}{20}, \text{\&c.},$$

are equal, the value of each being $\tfrac{1}{2}$. Also

$$\tfrac{1}{3}, \tfrac{2}{6}, \tfrac{3}{9}, \tfrac{4}{12}, \tfrac{5}{15}, \tfrac{6}{18}, \tfrac{7}{21}, \tfrac{8}{24}, \tfrac{9}{27}, \tfrac{10}{30}, \text{\&c.},$$

are equal fractions, the value of each of which is $\tfrac{1}{3}$. The fractions.

$$\tfrac{2}{3}, \tfrac{4}{6}, \tfrac{8}{12}, \tfrac{10}{15}, \tfrac{12}{18}, \tfrac{14}{21}, \tfrac{16}{24}, \text{\&c.},$$

have likewise all the same value; and lastly, we may conclude in general, that the fraction $\frac{a}{b}$ may be represented by the following expressions, each of which is equal to $\frac{a}{b}$; namely,

$$\frac{a}{b}, \frac{2a}{2b}, \frac{3a}{3b}, \frac{4a}{4b}, \frac{5a}{5b}, \frac{6a}{6b}, \frac{7a}{7b}, \text{\&c.}$$

87. To be convinced of this we have only to write for the value of the fraction $\frac{a}{b}$ a certain letter c, representing by this letter c the quotient of the division of a by b; and to recollect that the multiplication of the quotient c by the divisor b must give the dividend. For since c multiplied by b gives a, it is evident that c multiplied by $2b$ will give $2a$, that c multiplied by $3b$ will give

$3a$, and that in general c multiplied by mb must give ma. Now changing this into an example of division, and dividing the product ma, by mb one of the factors, the quotient must be equal to the other factor c; but ma divided by mb gives also the fraction $\frac{ma}{mb}$, which is consequently equal to c; and this is what was to be proved: for c having been assumed as the value of the fraction $\frac{a}{b}$, it is evident that this fraction is equal to the fraction $\frac{ma}{mb}$, whatever be the value of m.

88. We have seen that *every fraction may be represented in an infinite number of forms,* each of which contains the same value; and it is evident that of all these forms, that, which shall be composed of the least numbers, will be most easily understood. For example, we might substitute instead of $\frac{2}{3}$ the following fractions,

$$\tfrac{4}{6},\ \tfrac{6}{9},\ \tfrac{8}{12},\ \tfrac{10}{15},\ \tfrac{12}{18},\ \&c.;$$

but of all these expressions $\frac{2}{3}$ is that of which it is easiest to form an idea. Here therefore a problem arises, how a fraction, such as $\frac{8}{12}$, which is not expressed by the least possible numbers, may be reduced to its simplest form, or to *its least terms,* that is to say, in our present example, to $\frac{2}{3}$.

89. It will be easy to resolve this problem, if we consider that a fraction still preserves its value, when we multiply both its terms, or its numerator and denominator, by the same number. For from this it follows also, that *if we divide the numerator and denominator of a fraction by the same number, the fraction still preserves the same value.* This is made more evident by means of the general expression $\frac{ma}{mb}$; for if we divide both the numerator ma and the denominator mb by the number m, we obtain the fraction $\frac{a}{b}$, which, as was before proved, is equal to $\frac{ma}{mb}$.

90. In order therefore to reduce a given fraction to its least terms, it is required to find a number by which both the numerator and denominator may be divided. Such a number is called a *common divisor,* and so long as we can find a common divisor to the numerator and the denominator, it is certain that the fraction may be reduced to a lower form; but, on the con-

trary, when we see that except unity no other common divisor can be found, this shews that the fraction is already in the simplest form that it admits of.

91. To make this more clear, let us consider the fraction $\frac{48}{120}$. We see immediately that both the terms are divisible by 2, and that there results the fraction $\frac{24}{60}$. Then that it may again be divided by 2, and reduced to $\frac{12}{30}$; and this also, having 2 for a common divisor, it is evident, may be reduced to $\frac{6}{15}$. But now we easily perceive, that the numerator and denominator are still divisible by 3; performing this division, therefore, we obtain the fraction $\frac{2}{5}$, which is equal to the fraction proposed, and gives the simplest expression to which it can be reduced; for 2 and 5 have no common divisor but 1, which cannot diminish these numbers any further.

92. This property of fractions preserving an invariable value, whether we divide or multiply the numerator and denominator by the same number, is of the greatest importance, and is the principal foundation of the doctrine of fractions. For example, we can scarcely add together two fractions, or subtract them from each other, before we have, by means of this property, reduced them to other forms, that is to say, to expressions whose denominators are equal. Of this we shall treat in the following chapter.

93. We conclude the present by remarking, that all integers may also be represented by fractions. For example, 6 is the same as $\frac{6}{1}$, because 6 divided by 1 makes 6; and we may, in the same manner, express the number 6 by the fractions $\frac{12}{2}$, $\frac{18}{3}$, $\frac{24}{4}$, $\frac{36}{6}$, and an infinite number of others, which have the same value.

CHAPTER IX.

Of the Addition and Subtraction of Fractions.

94. When fractions have equal denominators, there is no difficulty in adding and subtracting them; for $\frac{2}{7}+\frac{3}{7}$ is equal to $\frac{5}{7}$, and $\frac{4}{7}-\frac{2}{7}$ is equal to $\frac{2}{7}$. In this case, either for addition or

subtraction, we alter only the numerators, and place the common denominator under the line; thus,

$\frac{7}{100}+\frac{9}{100}-\frac{12}{100}-\frac{15}{100}+\frac{20}{100}$ is equal to $\frac{9}{100}$; $\frac{24}{50}-\frac{7}{50}-\frac{12}{50}+\frac{31}{50}$ is equal to $\frac{36}{50}$, or $\frac{18}{25}$; $\frac{16}{20}-\frac{3}{20}-\frac{11}{20}+\frac{14}{20}$ is equal to $\frac{16}{20}$, or $\frac{4}{5}$; also $\frac{1}{3}+\frac{2}{3}$ is equal to $\frac{3}{3}$, or 1, that is to say, an integer; and $\frac{2}{4}-\frac{3}{4}+\frac{1}{4}$ is equal to $\frac{0}{4}$, that is to say, nothing, or 0.

95. But *when fractions have not equal denominators, we can always change them into other fractions that have the same denominator.* For example, when it is proposed to add together the fractions $\frac{1}{2}$ and $\frac{1}{3}$, we must consider that $\frac{1}{2}$ is the same as $\frac{3}{6}$, and that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$; we have therefore, instead of the two fractions proposed, these $\frac{3}{6}+\frac{2}{6}$, the sum of which is $\frac{5}{6}$. If the two fractions were united by the sign *minus*, as $\frac{1}{2}-\frac{1}{3}$, we should have $\frac{3}{6}-\frac{2}{6}$ or $\frac{1}{6}$.

Another example: let the fractions proposed be $\frac{3}{4}+\frac{5}{8}$; since $\frac{3}{4}$ is the same as $\frac{6}{8}$, this value may be substituted for it, and we may say $\frac{6}{8}+\frac{5}{8}$ makes $\frac{11}{8}$, or $1\frac{3}{8}$.

Suppose further, that the sum of $\frac{1}{3}$ and $\frac{1}{4}$ were required. I say that it is $\frac{7}{12}$; for $\frac{1}{3}$ makes $\frac{4}{12}$, and $\frac{1}{4}$ makes $\frac{3}{12}$.

96. *We may have a greater number of fractions to be reduced to a common denominator;* for example, $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$; *in this case the whole depends on finding a number which may be divisible by all the denominators of these fractions.* In this instance 60 is the number which has that property, and which consequently becomes the common denominator. We shall therefore have $\frac{30}{60}$ instead of $\frac{1}{2}$; $\frac{40}{60}$ instead of $\frac{2}{3}$; $\frac{45}{60}$ instead of $\frac{3}{4}$; $\frac{48}{60}$ instead of $\frac{4}{5}$; and $\frac{50}{60}$ instead of $\frac{5}{6}$. *If now it be required to add together* all these *fractions* $\frac{30}{60}, \frac{40}{60}, \frac{45}{60}, \frac{48}{60}$, and $\frac{50}{60}$, *we have only to add all the numerators, and under the sum place the common denominator* 60; that is to say, we shall have $\frac{213}{60}$, or three integers, and $\frac{33}{60}$, or $3\frac{11}{20}$.

97. The whole of this operation consists, as we before stated, in changing two fractions, whose denominators are unequal, into two others, whose denominators are equal. In order therefore to perform it generally, let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions proposed. First, multiply the two terms of the first fraction by d, we shall have the fraction $\frac{ad}{bd}$ equal to $\frac{a}{b}$; next multiply the two

terms of the second fraction by b, and we shall have an equivalent value of it expressed by $\frac{bc}{bd}$; thus the two denominators become equal. Now if the sum of the two proposed fractions be required, we may immediately answer that it is $\frac{ad+bc}{bd}$; and if their difference be asked, we say that it is $\frac{ad-bc}{bd}$. If the fractions $\frac{5}{8}$ and $\frac{7}{9}$, for example, were proposed, we should obtain in their stead $\frac{45}{72}$ and $\frac{56}{72}$; of which the sum is $\frac{101}{72}$, and the difference $\frac{11}{72}$.

98. To this part of the subject belongs also the question, of two proposed fractions, which is the greater or the less; for, to resolve this, we have only to reduce the two fractions to the same denominator. Let us take, for example, the two fractions $\frac{2}{3}$ and $\frac{5}{7}$: when reduced to the same denominator, the first becomes $\frac{14}{21}$, and the second $\frac{15}{21}$, and it is evident that the second, or $\frac{5}{7}$, is the greater, and exceeds the former by $\frac{1}{21}$.

Again, let the two fraction $\frac{3}{5}$ and $\frac{5}{8}$ be proposed. We shall have to substitute for them, $\frac{24}{40}$ and $\frac{25}{40}$; whence we may conclude that $\frac{5}{8}$ exceeds $\frac{3}{5}$, but only by $\frac{1}{40}$.

99. *When it is required to subtract a fraction from an integer,* it is sufficient to *change one of the units of that integer into a fraction having the same denominator as the fraction to be subtracted;* in the rest of the operation there is no difficulty. If it be required, for example, to subtract $\frac{2}{3}$ from 1, we write $\frac{3}{3}$ instead of 1, and say that $\frac{2}{3}$ taken from $\frac{3}{3}$ leaves the remainder $\frac{1}{3}$. So $\frac{5}{12}$, subtracted from 1, leaves $\frac{7}{12}$.

If it were required to subtract $\frac{3}{4}$ from 2, we should write 1 and $\frac{4}{4}$ instead of 2, and we should immediately see that after the subtraction there must remain $1\frac{1}{4}$.

100. It happens also sometimes, that having added two or more fractions together, we obtain more than an integer; that is to say, a numerator greater than the denominator: this is a case which has already occurred, and deserves attention.

We found, for example, article 96, that the sum of the five fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$, was $\frac{213}{60}$, and we remarked that the value of this sum was 3 integers and $\frac{33}{60}$, or $\frac{11}{20}$. Likewise $\frac{2}{3}+\frac{3}{4}$, or $\frac{8}{12}+\frac{9}{12}$, makes $\frac{17}{12}$, or $1\frac{5}{12}$. We have only to perform the actual division of the numerator by the denominator, to see how many integers there are for the quotient, and to set down the remainder. Nearly the same must be done to add together numbers compounded of integers and fractions; we first add the fractions, and if their sum produces one or more integers, these are added to the other integers. Let it be proposed, for example, to add $5\frac{1}{2}$ and $2\frac{2}{3}$; we first take the sum of $\frac{1}{2}$ and $\frac{2}{3}$, or of $\frac{3}{6}$ and $\frac{4}{6}$. It is $\frac{7}{6}$ or $1\frac{1}{6}$; then the sum total is $6\frac{1}{6}$.

CHAPTER X.

Of the Multiplication and Division of Fractions.

101. THE *rule for the multiplication of a fraction by an integer, or whole number, is to multiply the numerator only by the given number, and not to change the denominator*: thus,

2 times, or twice $\frac{1}{2}$ makes $\frac{2}{2}$, or 1 integer;
2 times, or twice $\frac{1}{3}$ makes $\frac{2}{3}$;
3 times, or thrice $\frac{1}{6}$ makes $\frac{3}{6}$, or $\frac{1}{2}$; and
4 times $\frac{5}{12}$ makes $\frac{20}{12}$ or $1\frac{8}{12}$, or $1\frac{2}{3}$.

But, instead of this rule, we may use that of dividing the denominator by the given integer; and this is preferable, when it can be used, because it shortens the operation. Let it be required, for example, to multiply $\frac{8}{9}$ by 3; if we multiply the numerator by the given integer we obtain $\frac{24}{9}$, which product we must reduce to $\frac{8}{3}$. But if we do not change the numerator, and divide the denominator by the integer, we find immediately $\frac{8}{3}$, or $2\frac{2}{3}$ for the given product. Likewise $\frac{13}{24}$ multiplied by 6 gives $\frac{13}{4}$, or $3\frac{1}{4}$.

102. In general, therefore, the product of the multiplication of a fraction $\frac{a}{b}$ by c is $\frac{ac}{b}$; and it may be remarked, *when the integer is exactly equal to the denominator*, that *the product must be equal to the numerator.*

$$\text{So that} \begin{cases} \frac{1}{2} \text{ taken twice gives } 1; \\ \frac{2}{3} \text{ taken thrice gives } 2; \\ \frac{3}{4} \text{ taken 4 times gives } 3. \end{cases}$$

And in general, if we multiply the fraction $\frac{a}{b}$ by the number b, the product must be a, as we have already shewn; for since

$\frac{a}{b}$ expresses the quotient resulting from the division of the dividend a by the divisor b, and since it has been demonstrated that the quotient multiplied by the divisor will give the dividend, it is evident that $\frac{a}{b}$ multiplied by b must produce a.

103. We have shewn how a fraction is to be multiplied by an integer; let us now consider also *how a fraction is to be divided by an integer;* this inquiry is necessary before we proceed to the multiplication of fractions by fractions. It is evident, if I have to divide the fraction $\frac{2}{3}$ by 2, that the result must be $\frac{1}{3}$; and that the quotient of $\frac{6}{7}$ divided by 3 is $\frac{2}{7}$. The rule therefore is, to *divide the numerator by the integer without changing the denominator.* Thus,

$\frac{12}{25}$ divided by 2 gives $\frac{6}{25}$;
$\frac{12}{25}$ divided by 3 gives $\frac{4}{25}$; and
$\frac{12}{25}$ divided by 4 gives $\frac{3}{25}$; &c.

104. This rule may be easily practised, provided the numerator be divisible by the number proposed; but very often it is not: it must therefore be observed that a fraction may be transformed into an infinite number of other expressions, and in that number there must be some by which the numerator might be divided by the given integer. If it were required, for example, to divide $\frac{3}{4}$ by 2, we should change the fraction into $\frac{6}{8}$, and then dividing the numerator by 2, we should immediately have $\frac{3}{8}$ for the quotient sought.

In general, if it be proposed to divide the fraction $\frac{a}{b}$ by c, we change it into $\frac{ac}{bc}$, and then dividing the numerator ac by c, write $\frac{a}{bc}$ for the quotient sought.

105. *When therefore a fraction* $\frac{a}{b}$ *is to be divided by an integer* c, *we have only to multiply the denominator by that number, and leave the numerator as it is.* Thus $\frac{5}{8}$ divided by 3 gives $\frac{5}{24}$, and $\frac{9}{6}$ divided by 5 gives $\frac{9}{30}$.

This operation becomes easier when the numerator itself is divisible by the integer, as we have supposed in article 103.

For example, $\frac{9}{16}$ divided by 3 would give, according to our last rule, $\frac{9}{48}$; but by the first rule, which is applicable here, we obtain $\frac{3}{16}$, an expression equivalent to $\frac{9}{48}$, but more simple.

106. We shall now be able to understand how one fraction $\frac{a}{b}$ may be multiplied by another fraction $\frac{c}{d}$. We have only to consider that $\frac{c}{d}$ means that c is divided by d; and on this principle, we shall first multiply the fraction $\frac{a}{b}$ by c, which produces the result $\frac{ac}{b}$; after which we shall divide by d, which gives $\frac{ac}{bd}$.

Hence the following rule for multiplying fractions; multiply separately the numerators and the denominators.

Thus $\frac{1}{2}$ by $\frac{2}{3}$ gives the product $\frac{2}{6}$, or $\frac{1}{3}$;
$\frac{2}{3}$ by $\frac{4}{5}$ makes $\frac{8}{15}$;
$\frac{3}{4}$ by $\frac{5}{12}$ produces $\frac{15}{48}$, or $\frac{5}{16}$; &c.

107. It remains to shew *how one fraction may be divided by another.* We remark first, that *if the two fractions have the same number for a denominator, the division takes place only with respect to the numerators;* for it is evident, that $\frac{3}{12}$ is contained as many times in $\frac{9}{12}$ as 3 in 9, that is to say, thrice; and in the same manner, in order to divide $\frac{8}{12}$ by $\frac{9}{12}$, we have only to divide 8 by 9, which gives $\frac{8}{9}$. We shall also have $\frac{6}{20}$ in $\frac{18}{20}$, 3 times: $\frac{7}{100}$ in $\frac{49}{100}$, 7 times; $\frac{7}{25}$ in $\frac{6}{25}$, $\frac{6}{7}$; &c.

108. But *when the fractions have not equal denominators,* we must have recourse to the method already mentioned for reducing them to a common denominator. Let there be, for example, the fraction $\frac{a}{b}$ to be divided by the fraction $\frac{c}{d}$; we first reduce them to the same denominator; we have then $\frac{ad}{bd}$ to be divided by $\frac{bc}{bd}$; it is now evident, that the quotient must be represented simply by the division of ad by bc; which gives $\frac{ad}{bc}$.

Hence the following rule : *Multiply the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor ; the first product will be the numerator of the quotient, and the second will be its denominator.*

109. Applying this rule to the division of $\frac{5}{8}$ by $\frac{2}{3}$, we shall have the quotient $\frac{15}{16}$; the division of $\frac{3}{4}$ by $\frac{1}{2}$ will give $\frac{6}{4}$ or $\frac{3}{2}$ or 1 and $\frac{1}{2}$; and $\frac{25}{48}$ by $\frac{5}{6}$ will give $\frac{150}{240}$, or $\frac{5}{8}$.

110. This rule for division is often represented in a manner more easily remembered, as follows : *Invert the fraction which is the divisor, so that the denominator may be in the place of the numerator, and the latter be written under the line ; then multiply the fraction, which is the dividend by this inverted fraction, and the product will be the quotient sought.* Thus $\frac{3}{4}$ divided by $\frac{1}{2}$ is the same as $\frac{3}{4}$ multiplied by $\frac{2}{1}$, which makes $\frac{6}{4}$, or $1\frac{1}{2}$. Also $\frac{5}{8}$ divided by $\frac{2}{3}$ is the same as $\frac{5}{8}$ multiplied by $\frac{3}{2}$, which is $\frac{15}{16}$; or $\frac{25}{48}$ divided by $\frac{5}{6}$ gives the same $\frac{25}{48}$ multiplied by $\frac{6}{5}$, the product of which is $\frac{150}{240}$, or $\frac{5}{8}$.

We see then, in general, that *to divide by the fraction $\frac{1}{2}$, is the same as to multiply by $\frac{2}{1}$*, or 2 *; that division by $\frac{1}{3}$ amounts to multiplication by $\frac{3}{1}$, or by 3, &c.*

111. The number 100 divided by $\frac{1}{2}$ will give 200 ; and 1000 divided $\frac{1}{3}$ will give 3000. Further, if it were required to divide 1 by $\frac{1}{1000}$, the quotient would be 1000 ; and dividing 1 by $\frac{1}{100000}$, the quotient is 100000. This enables us to conceive that, when any number is divided by 0, the result must be a number infinitely great ; for even the division of 1 by the small fraction $\frac{1}{1000000000}$ gives for the quotient the very great number 1000000000.

112. Every number when divided by itself producing unity, it is evident that a fraction divided by itself must also give 1 for the quotient. The same follows from our rule : for, in order to divide $\frac{3}{4}$ by $\frac{3}{4}$, we must multiply $\frac{3}{4}$ by $\frac{4}{3}$, and we obtain $\frac{12}{12}$, or 1 ; and if it be required to divide $\frac{a}{b}$ by $\frac{a}{b}$, we multiply $\frac{a}{b}$ by $\frac{b}{a}$; now the product $\frac{a\,b}{a\,b}$ is equal to 1.

113. We have still to explain an expression which is frequently used. It may be asked, for example, what is the half of $\frac{3}{4}$; this means that we must multiply $\frac{3}{4}$ by $\frac{1}{2}$. So likewise, if the value of $\frac{2}{3}$ of $\frac{5}{8}$ were required, we should multiply $\frac{5}{8}$ by $\frac{2}{3}$, which produces $\frac{10}{24}$; and $\frac{3}{4}$ of $\frac{9}{16}$ is the same as $\frac{9}{16}$ multiplied by $\frac{3}{4}$, which produces $\frac{27}{64}$.

114. Lastly, we must here observe the same rules with respect to the signs $+$ and $-$, that we before laid down for integers. Thus $+\frac{1}{2}$ multiplied by $-\frac{1}{3}$ makes $-\frac{1}{6}$; and $-\frac{2}{3}$ multipled by $-\frac{4}{5}$ gives $+\frac{8}{15}$. Farther, $-\frac{5}{8}$ divided by $+\frac{2}{3}$ makes $-\frac{15}{16}$; and $-\frac{3}{4}$ divided by $-\frac{3}{4}$ makes $+\frac{12}{12}$ or $+1$.

CHAPTER XI.

Of Square Numbers.

115. THE *product of a number, when multiplied by itself, is called a square; and for this reason, the number, considered in relation to such a product, is called a square root.*

For example, when we multiply 12 by 12, the product 144 is a square, of which the root is 12.

This term is derived from geometry, which teaches us that the contents of a square are found by multiplying its side by itself.

116. Square numbers are found therefore by multiplication; that is to say, by multiplying the root by itself. Thus 1 is the square of 1, since 1 multiplied by 1 makes 1; likewise, 4 is the square of 2; and 9 the square of 3; 2 also is the root of 4, and 3 is the root of 9.

We shall begin by considering the squares of natural numbers, and shall first give the following small table, on the first line of which several numbers, or roots, are placed, and on the second their squares.

Numbers.	1	2	3	4	5	6	7	8	9	10	11	12	13
Squares.	1	4	9	16	25	36	49	64	81	100	121	144	169

117. It will be readily perceived, that the series of square numbers thus arranged has a singular property; namely, that if each of them be subtracted from that which immediately follows, the remainders always increase by 2, and form this series;

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c.

118. *The squares of fractions are found in the same manner, by multiplying any given fraction by itself.* For example, the square of $\frac{1}{2}$ is $\frac{1}{4}$,

$$\text{The square of} \left\{ \begin{matrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{4} \\ \frac{3}{4} \end{matrix} \right\} \text{is} \left\{ \begin{matrix} \frac{1}{9}; \\ \frac{4}{9}; \\ \frac{1}{16}; \\ \frac{9}{16}, \text{ \&c.} \end{matrix} \right.$$

We have only therefore to divide the square of the numerator by the square of the denominator, and the fraction, which expresses that division, must be the square of the given fraction. Thus, $\frac{25}{64}$ is the square of $\frac{5}{8}$; and reciprocally, $\frac{5}{8}$ is the root of $\frac{25}{64}$.

119. When the square of a mixed number, or a number, composed of an integer and a fraction, is required, we have only to reduce it to a single fraction, and then to take the square of that fraction. Let it be required, for example, to find the square of $2\frac{1}{2}$; we first express this number by $\frac{5}{2}$, and taking the square of that fraction, we have $\frac{25}{4}$, or $6\frac{1}{4}$, for the value of the square of $2\frac{1}{2}$. So to obtain the square of $3\frac{1}{4}$, we say $3\frac{1}{4}$ is equal to $\frac{13}{4}$; therefore its square is equal to $\frac{169}{16}$, or to 10 and $\frac{9}{16}$. The squares of the numbers between 3 and 4, supposing them to increase by one fourth, are as follows:

Numbers.	3	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4
Squares.	9	$10\frac{9}{16}$	$12\frac{1}{4}$	$14\frac{1}{16}$	16

From this small table we may infer, that if a root contain a fraction, its square also contains one. Let the root, for example, be $1\frac{5}{12}$; its square is $\frac{289}{144}$, or $2\frac{1}{144}$; that is to say, a little greater than the integer 2.

120. Let us proceed of general expressions. When the root is a, the square must be aa; if the root be $2a$, the square is $4aa$;

which shews that by doubling the root, the square becomes 4 times greater. So if the root be $3a$, the square is $9aa$; and if the root be $4a$, the square is $16aa$. But if the root be ab, the square is $aabb$; and if the root be abc, the square is $aabbcc$.

121. Thus *when the root is composed of two, or more factors, we multiply their squares together; and reciprocally, if a square be composed of two or more factors, of which each is a square, we have only to multiply together the roots of those squares, to obtain the complete root of the square proposed.* Thus, as 2304 is equal to $4 \times 16 \times 36$, the square root of it is $2 \times 4 \times 6$, or 48; and 48 is found to be the true square root of 2304, because 48×48 gives 2304.

122. Let us now consider what rule is to be observed with regard to the signs $+$ and $-$. First, it is evident that if the root has the sign $+$, that is to say, is a positive number, its square must necessarily be a positive number also, because $+$ by $+$ makes $+$: the square of $+a$ will be $+aa$. But if the root be a negative number, as $-a$, the square is still positive, for it is $+aa$; we may therefore conclude, that $+aa$ *is the square both of* $+$a *and of* $-$a, and that *consequently every square has two roots, one positive and the other negative.* The square root of 25, for example, is both $+5$ and -5, because -5 multiplied by -5 gives 25, as well as $+5$ by $+5$.

CHAPTER XII.

Of Square Roots, and of Irrational Numbers resulting from them.

123. What we have said in the preceding chapter is chiefly this: that the square root of a given number is nothing but a number whose square is equal to the given number; and that we may put before these roots either the positive or the negative sign.

124. So that when a square number is given, provided we retain in our memory a sufficient number of square numbers, it is easy to find its root. If 196, for example, be the given number, we know that its square root is 14.

Fractions likewise are easily managed: it is evident, for example, that $\frac{5}{7}$ is the square root of $\frac{25}{49}$. To be convinced of this, we have only to take the square root of the numerator, and that of the denominator.

If the number proposed be a mixed number, as $12\frac{1}{4}$, we reduce it to a single fraction, which here is $\frac{49}{4}$, and we immediately perceive that $\frac{7}{2}$, or $3\frac{1}{2}$, must be the square root of $12\frac{1}{4}$.

125. But when the given number is not a square, as 12, for example, it is not possible to extract its square root: or to find a number, which, multiplied by itself, will give the product 12. We know, however, that the square root of 12 must be greater than 3, because 3×3 produces only 9: and less than 4, because 4×4 produces 16, which is more than 12. We know also, that this root is less than $3\frac{1}{2}$; for we have seen that the square of $3\frac{1}{2}$, or $\frac{7}{2}$ is $12\frac{1}{4}$. Lastly, we may approach still nearer to this root, by comparing it with $3\frac{7}{15}$; for the square of $3\frac{7}{15}$, or of $\frac{52}{15}$ is $\frac{2704}{225}$, or $12\frac{4}{225}$, so that this fraction is still greater than the root required; but very little greater, as the difference of the two sqares is only $\frac{4}{225}$.

126. We may suppose that as $3\frac{1}{2}$ and $3\frac{7}{15}$ are numbers greater than the root of 12, it might be possible to add to 3 a fraction a little less than $\frac{7}{15}$, and precisely such that the square of the sum would be equal to 12.

Let us therefore try with $3\frac{3}{7}$, since $\frac{3}{7}$ is a little less than $\frac{7}{15}$. Now $3\frac{3}{7}$ is equal to $\frac{24}{7}$, the square of which is $\frac{576}{49}$, and consequently less by $\frac{12}{49}$ than 12, which may be expressed by $\frac{588}{49}$. It is therefore proved that $3\frac{3}{7}$ is less, and that $3\frac{7}{15}$ is greater than the root required. Let us then try a number a little greater than $3\frac{3}{7}$, but yet less than $3\frac{7}{15}$, for example, $3\frac{5}{11}$. This number, which is equal to $\frac{38}{11}$, has for its square $\frac{1444}{121}$. Now, by reducing 12 to this denominator, we obtain $\frac{1452}{121}$; which shows that $3\frac{5}{11}$ is still less than the root of 12, viz. by $\frac{8}{121}$. Let us therefore substitute for $\frac{5}{11}$ the fraction $\frac{6}{13}$, which is a little greater, and see what will be the result of the comparison of the square of $3\frac{6}{13}$ with the proposed number 12. The square of $3\frac{6}{13}$ is $\frac{2025}{169}$; now 12 reduced to the same denominator is $\frac{2028}{169}$; so that $3\frac{6}{13}$ is still too small, though only by $\frac{3}{169}$, whilst $3\frac{7}{15}$ has been found too great.

127. It is evident therefore, that whatever fraction be joined to 3, the square of that sum must always contain a fraction, and can never be exactly equal to the integer 12. Thus, although we know that the square root of 12 is greater than $3\frac{6}{13}$ and less than $3\frac{7}{13}$, yet we are unable to assign an intermediate fraction between these two, which, at the same time, if added to 3, would express exactly the square root of 12. Notwithstanding this, we are not to assert that the square root of 12 is absolutely and in itself indeterminate; it only follows from what has been said, that this root, though it necessarily has a determinate magnitude, cannot be expressed by fractions.

128. *There is therefore a sort of numbers which cannot be assigned by fractions, and which are nevertheless determinate quantities;* the square root of 12 furnishes an example. We call this new species of numbers, *irrational numbers;* they occur whenever we endeavour to find the square root of a number which is not a square. Thus, 2 not being a perfect square, the square root of 2, or the number which, multiplied by itself, would produce 2, is an irrational quantity. These numbers are also called *surd quantities*, or *incommensurables.*

129. These irrational quantities, though they cannot be expressed by fractions, are nevertheless magnitudes, of which we may form an accurate idea. For however concealed the square root of 12, for example, may appear, we are not ignorant, that it must be a number which, when multiplied by itself, would exactly produce 12; and this property is sufficient to give us an idea of the number, since it is in our power to approximate its value continually.

130. As we are therefore sufficiently acquainted with the nature of the irrational numbers, under our present consideration, a particular sign has been agreed on, to express the square roots of all numbers that are not perfect squares. This sign is written thus $\surd$, and is read *square root.* Thus, $\sqrt{12}$ represents the square root of 12, or the number which, multiplied by itself, produces 12. So, $\sqrt{2}$ represents the square root of 2; $\sqrt{3}$ that of 3; $\sqrt{\frac{2}{3}}$ that of $\frac{2}{3}$ and, in general, $\sqrt{a}$ *represents the square root of the number* a. Whenever therefore we would express the

square root of a number which is not a square, we need only make use of the mark $\surd$ by placing it before the number.

131. The explanation, which we have given of irrational numbers, will readily enable us to apply to them the known methods of calculation. For knowing that the square root of 2, multiplied by itself, must produce 2; we know also, that the multiplication $\sqrt{2}$ by $\sqrt{2}$ must necessarily produce 2; that, in the same manner, the multiplication of $\sqrt{3}$ by $\sqrt{3}$ must give 3: that $\sqrt{5}$ by $\sqrt{5}$ makes 5; that $\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{2}{3}}$ makes $\frac{2}{3}$; and, in general, that *$\sqrt{a}$ multiplied by $\sqrt{a}$ produces* a.

132. *But when it is required to multiply $\sqrt{a}$ by $\sqrt{b}$ the product will be found to be $\sqrt{ab}$*; because we have shewn before, that if a square has two or more factors, its root must be composed of the roots of those factors. Wherefore we find the square root of the product $a\,b$, which is $\sqrt{ab}$, by multiplying the square root of a or $\sqrt{a}$, by the square root of b or $\sqrt{b}$. It is evident from this, that if b were equal to a, we should have $\sqrt{aa}$ for the product of $\sqrt{a}$ by $\sqrt{b}$. Now $\sqrt{aa}$ is evidently a, since $a\,a$ is the square of a.

133. In division, if it were required to divide $\sqrt{a}$, for example, by $\sqrt{b}$, we obtain $\sqrt{\frac{a}{b}}$; and in this instance the irrationality may vanish in the quotient. Thus, having to divide $\sqrt{18}$ by $\sqrt{8}$, the quotient is $\sqrt{\frac{18}{8}}$, which is reduced to $\sqrt{\frac{9}{4}}$, and consequently to $\frac{3}{2}$, because $\frac{9}{4}$ is the square of $\frac{3}{2}$.

134. When the number, before which we have placed the radical sign $\surd$, is itself a square, its root is expressed in the usual way. Thus $\sqrt{4}$ is the same as 2; $\sqrt{9}$ the same as 3; $\sqrt{36}$ the same as 6; and $\sqrt{12\frac{1}{4}}$ the same as $\frac{7}{2}$, or $3\frac{1}{2}$. In these instances the irrationality is only apparent, and vanishes of course.

135. It is easy also to multiply irrational numbers by ordinary numbers. For example, 2 *multiplied by $\sqrt{5}$ makes* $2\sqrt{5}$, and 3 times $\sqrt{2}$ make $3\sqrt{2}$. In the second example, however, as 3 is equal to $\sqrt{9}$, we may also express 3 times $\sqrt{2}$ by $\sqrt{9}$ times $\sqrt{2}$, or by $\sqrt{18}$. So $2\sqrt{a}$ is the same as $\sqrt{4a}$, and $3\sqrt{a}$ the same as $\sqrt{9a}$. And, in general, *$b\sqrt{a}$ has the same value as the square root of* b b a, *or* $\sqrt{abb}$; whence we infer reciprocally, that when the number which is preceded by the radical

sign contains a square, we may take the root of that square and put it before the sign, as we should do in writing $b\sqrt{a}$ instead of $\sqrt{abb}$. After this, the following reductions will be easily understood:

$$\left.\begin{array}{ll}\sqrt{8}, & \text{or } \sqrt{2\cdot 4}\\ \sqrt{12}, & \text{or } \sqrt{3\cdot 4}\\ \sqrt{18}, & \text{or } \sqrt{2\cdot 9}\\ \sqrt{24}, & \text{or } \sqrt{6\cdot 4}\\ \sqrt{32}, & \text{or } \sqrt{2\cdot 16}\\ \sqrt{75}, & \text{or } \sqrt{3\cdot 25}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}2\sqrt{2};\\ 2\sqrt{3};\\ 3\sqrt{2};\\ 2\sqrt{6};\\ 4\sqrt{2};\\ 5\sqrt{3};\end{array}\right.$$

and so on.

136. Division is founded on the same principles. *$\sqrt{a}$ divided by $\sqrt{b}$, gives $\frac{\sqrt{a}}{\sqrt{b}}$, or $\sqrt{\frac{a}{b}}$.* In the same manner,

$$\left.\begin{array}{l}\frac{\sqrt{8}}{\sqrt{2}}\\[2ex] \frac{\sqrt{18}}{\sqrt{2}}\\[2ex] \frac{\sqrt{12}}{\sqrt{3}}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}\sqrt{\frac{8}{2}}, \text{ or } \sqrt{4}, \text{ or } 2;\\[2ex] \sqrt{\frac{18}{2}}, \text{ or } \sqrt{9}, \text{ or } 3;\\[2ex] \sqrt{\frac{12}{3}}, \text{ or } \sqrt{4}, \text{ or } 2.\end{array}\right.$$

Further
$$\left.\begin{array}{l}\frac{2}{\sqrt{2}}\\[2ex] \frac{3}{\sqrt{3}}\\[2ex] \frac{12}{\sqrt{6}}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}\frac{\sqrt{4}}{\sqrt{2}}, \text{ or } \sqrt{\tfrac{4}{2}}, \text{ or } \sqrt{2};\\[2ex] \frac{\sqrt{9}}{\sqrt{3}}, \text{ or } \sqrt{\tfrac{9}{3}}, \text{ or } \sqrt{3};\\[2ex] \frac{\sqrt{144}}{\sqrt{6}}, \text{ or } \sqrt{\tfrac{144}{6}}, \text{ or } \sqrt{24},\end{array}\right.$$

or $\sqrt{6\cdot 4}$, or lastly $2\sqrt{6}$.

137. There is nothing in particular to be observed with respect to the addition and subtraction of such quantities, because we only connect them by the signs $+$ and $-$. For example, *$\sqrt{2}$ added to $\sqrt{3}$ is written $\sqrt{2}+\sqrt{3}$; and $\sqrt{3}$ subtracted from $\sqrt{5}$ is written $\sqrt{5}-\sqrt{3}$.*

138. We may observe lastly, that in order to distinguish irrational numbers, we call all other numbers, both integral and fractional, *rational numbers.*

So that, whenever we speak of rational numbers, we understand integers or fractions.

CHAPTER XIII.

Of Impossible or Imaginary Quantities, which arise from the same source.

139. We have already seen that the squares of numbers, negative as well as positive, are always positive, or affected with the sign $+$; having shewn that $-a$ multiplied by $-a$ gives $+aa$, the same as the product of $+a$ by $+a$. Wherefore, in the preceding chapter, we supposed that all the numbers, of which it was required to extract the square roots, were positive.

140. When it is required therefore to extract the root of a negative number, a very great difficulty arises; since there is no assignable number, the square of which would be a negative quantity. Suppose, for example, that we wished to extract the root of -4; we require such a number, as when multiplied by itself, would produce -4; now this number is neither $+2$ nor -2, because the square, both of $+2$ and of -2, is $+4$, and not -4.

141. We must therefore conclude, that *the square root of a negative number cannot be either a positive number, or a negative number*, since the squares of negative numbers also take the sign *plus.* Consequently the root in question must belong to an entirely distinct species of numbers; since it cannot be ranked either among positive, or among negative numbers.

142. Now, we before remarked, that positive numbers are all greater than nothing, or 0, and that negative numbers are all less than nothing, or 0; so that whatever exceeds 0, is expressed by positive numbers, and whatever is less than 0, is expressed by negative numbers. The square roots of negative numbers, therefore, are neither greater nor less than nothing. We can-

not say however, that they are 0; for 0 multiplied by 0 produces 0, and consequently does not give a negative number.

143. Now, since all numbers, which it is possible to conceive, are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number amongst possible numbers, and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers which from their nature are impossible. *These numbers are usually called imaginary quantities,* because they exist merely in the imagination.

144. All such expressions, as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, &c., are consequently impossible, or imaginary numbers, since they represent roots of negative quantities: and of such numbers we may truly assert, that they are neither nothing, nor greater than nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible.

145. But notwithstanding all this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number which, multiplied by itself, produces -4. For this reason also, nothing prevents us from making use of these imaginary numbers, and employing them in calculation.

146. The first idea that occurs on the present subject is, that the square of $\sqrt{-3}$, for example, or the product of $\sqrt{-3}$ by $\sqrt{-3}$, must be -3; that the product of $\sqrt{-1}$ by $\sqrt{-1}$ is -1; and, in general, that by multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or by taking the square of $\sqrt{-a}$, we obtain $-a$.

147. Now, as $-a$ is equal to $+a$ multiplied by -1, and as the square root of a product is found by multiplying together the roots of its factors, it follows that the root of a multiplied by -1, or $\sqrt{-a}$, is equal to $\sqrt{a}$ multiplied by $\sqrt{-1}$. Now $\sqrt{a}$ is a possible or real number, consequently *the whole impossibility of an imaginary quantity may be always reduced to* $\sqrt{-1}$. For this reason, $\sqrt{-4}$ is equal to $\sqrt{4}$ multiplied by $\sqrt{-1}$, and equal to $2\sqrt{-1}$, on account of $\sqrt{4}$ being equal to 2. For the same reason, $\sqrt{-9}$ is reduced to $\sqrt{9} \times \sqrt{-1}$, or $3\sqrt{-1}$; and $\sqrt{-16}$ is equal to $4\sqrt{-1}$.

148. Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$, or 2, for the value of the product of $\sqrt{-1}$ by $\sqrt{-4}$. We see, therefore, that *two imaginary numbers, multiplied together, produce a real, or possible one.*

But, on the contrary, *a possible number, multiplied by an impossible number, gives always an imaginary product:* thus, $\sqrt{-3}$ by $\sqrt{+5}$ gives $\sqrt{-15}$.

149. It is the same with regard to division; for $\sqrt{a}$ divided by $\sqrt{b}$ making $\sqrt{\frac{a}{b}}$, it is evident that $\sqrt{-4}$ divided by $\sqrt{-1}$ will make $\sqrt{+4}$, or 2; that $\sqrt{+3}$ divided by $\sqrt{-3}$ will give $\sqrt{-1}$; and that 1 divided by $\sqrt{-1}$ gives $\sqrt{\frac{+1}{-1}}$, or $\sqrt{-1}$; because 1 is equal to $\sqrt{+1}$.

150. We have before observed, that the square root of any number has always two values, one positive and the other negative; that $\sqrt{4}$, for example, is both $+2$ and -2, and that in general, we must take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a. This remark applies also to imaginary numbers; *the square root of* $-a$ *is both* $+\sqrt{-a}$ *and* $-\sqrt{-a}$; *but we must not confound the signs* $+$ *and* $-$, *which are before the radical sign* $\sqrt{}$, *with the sign which comes after it.*

151. It remains for us to remove any doubt which may be entertained concerning the utility of the numbers of which we have been speaking; for those numbers being impossible, it would not be surprising if any one should think them entirely useless, and the subject only of idle speculation. This however is not the case. The calculation of imaginary quantities is of the greatest importance: questions frequently arise, of which we cannot immediately say, whether they include any thing real and possible, or not. Now, when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible.*

* This is followed in the original by an example intended to illustrate what is here said. It is omitted by the Editor, as it implies a degree of acquaintance with the subject, which the learner cannot be supposed to possess at this stage of his progress.

CHAPTER XIV.

Of Cubic Numbers.

152. WHEN *a number has been multiplied twice by itself, or, which is the same thing, when the square of a number has been multiplied once more by that number, we obtain a product which is called a cube, or a cubic number.* Thus, the cube of a is $a\,a\,a$, since it is the product obtained by multiplying a by itself, or by a, and that square $a\,a$ again by a.

The cubes of the natural numbers therefore succeed each other in the following order.

Numbers.	1	2	3	4	5	6	7	8	9	10
Cubes.	1	8	27	64	125	216	343	512	729	1000

153. If we consider the differences of these cubes, as we did those of the squares, by subtracting each cube from that which comes after it, we shall obtain the following series of numbers :

7, 19, 37, 61, 91, 127, 169, 217, 271.

At first we do not observe any regularity in them ; but if we take the respective differences of these numbers, we find the following series :

12, 18, 24, 30, 36, 42, 48, 54, 60 ;

in which the terms, it is evident, increase always by 6.

154. After the definition we have given of a cube, it will not be difficult to find the cube of fractional numbers ; $\frac{1}{8}$ is the cube of $\frac{1}{2}$; $\frac{1}{27}$ is the cube of $\frac{1}{3}$; and $\frac{8}{27}$ is the cube of $\frac{2}{3}$. In the same manner, we have only to take the cube of the numerator and that of the denominator separately, and we shall have as the cube of $\frac{3}{4}$, for instance, $\frac{27}{64}$.

155. If it be required *to find the cube of a mixed number, we must first reduce it to a single fraction, and then proceed in the manner that has been described.* To find, for example, the cube of $1\frac{1}{2}$, we must take that of $\frac{3}{2}$, which is $\frac{27}{8}$, or 3 and $\frac{3}{8}$. So the cube of $1\frac{1}{4}$, or of the single fraction $\frac{5}{4}$, is $\frac{125}{64}$, or $1\frac{61}{64}$; and the cube of $3\frac{1}{4}$, or of $\frac{13}{4}$ is $\frac{2197}{64}$, or $34\frac{21}{64}$.

156. Since $a\,a\,a$ is the cube of a, that of $a\,b$ will be $a\,a\,a\,b\,b\,b$; whence we see, that *if a number has two or more factors, we may find its cube by multiplying together the cubes of those factors.* For example, as 12 is equal to 3×4, we multiply the cube of 3, which is 27, by the cube of 4, which is 64, and we obtain 1728, for the cube of 12. Further, the cube of $2\,a$ is $8\,a\,a\,a$, and consequently 8 times greater than the cube of a: and likewise, the cube of $3\,a$ is $27\,a\,a\,a$, that is to say, 27 times greater than the cube of a.

157. *Let us attend here also to the signs* $+$ *and* $-$. It is evident that the cube of a positive number $+a$ must also be positive, that is $+a\,a\,a$. But if it be required to cube a negative number $-a$, it is found by first taking the square, which is $+a\,a$, and then multiplying, according to the rule, this square by $-a$, which gives for the cube required $-a\,a\,a$. *In this respect,* therefore, *it is not the same with cubic numbers as with squares, since the latter are always positive: whereas the cube of* -1 *is* -1, *that of* -2 *is* -8, *that of* -3 *is* -27, *and so on.*

CHAPTER XV.

Of Cube Roots, and of irrational numbers resulting from them.

158. As we can, in the manner already explained, find the cube of a given number, so, when a number is proposed, we may also reciprocally find a number, which, multiplied twice by itself, will produce that number. The number here sought is called, with relation to the other, *the cube root. So that the cube root of a given number is the number whose cube is equal to that given number.*

159. It is easy therefore to determine the cube root, when the number proposed is a real cube, such as the examples in the last chapter. For we easily perceive that the cube root of 1 is 1; that of 8 is 2; that of 27 is 3; that of 64 is 4, and so on. And in the same manner, the cube root of -27 is -3; and that of -125 is -5.

Further, if the proposed number be a fraction, as $\frac{8}{27}$, the cube root of it must be $\frac{2}{3}$; and that of $\frac{64}{343}$ is $\frac{4}{7}$. Lastly, the cube root of a mixed number $2\frac{10}{27}$ must be $\frac{4}{3}$, or $1\frac{1}{3}$: because $2\frac{10}{27}$ is equal to $\frac{64}{27}$

160. But if the proposed number be not a cube, its cube root cannot be expressed either in integers, or in fractional numbers. For example, 43 is not a cubic number; I say therefore that it is impossible to assign any number, either integer or fractional, whose cube shall be exactly 43. We may however affirm, that the cube root of that number is greater than 3, since the cube of 3 is only 27; and less than 4, because the cube of 4 is 64. We know therefore, that the cube root required is necessarily contained between the numbers 3 and 4.

161. Since the cube root of 43 is greater than 3, if we add a fraction to 3, it is certain that we may approximate still nearer and nearer to the true value of this root: but we can never assign the number which expresses that value exactly; because the cube of a mixed number can never be perfectly equal to an integer, such as 43. If we were to suppose, for example, $3\frac{1}{2}$, or $\frac{7}{2}$ to be the cube root required, the error would be $\frac{1}{8}$; for the cube of $\frac{7}{2}$ is only $\frac{343}{8}$, or $42\frac{7}{8}$.

162. This therefore shews, that *the cube root of* 43 *cannot be expressed in any way, either by integers or by fractions.* However we have a distinct idea of the magnitude of this root; which induces us to use, in order to represent it, *the sign* $\sqrt[3]{}$, which we place before the proposed number, and which *is read cube root, to distinguish it from the square root, which is often called simply the root.* Thus $\sqrt[3]{43}$ means the cube root of 43, that is to say, the number whose cube is 43, or which, multiplied twice by itself, produces 43.

163. It is evident also, that such expressions cannot belong to rational quantities, and that they rather form a particular species of irrational quantities. They have nothing in common with square roots, and it is not possible to express such a cube root by a square root; as, for example, by $\sqrt{12}$; for the square

of $\sqrt{12}$ being 12, its cube will be $12\sqrt{12}$, consequently still irrational, and such cannot be equal to 43.

164. If the proposed number be a real cube, our expressions become rational; $\sqrt[3]{1}$ *is equal to* 1; $\sqrt[3]{8}$ *is equal to* 2; $\sqrt[3]{27}$ *is equal to* 3; *and, generally,* $\sqrt[3]{aaa}$ *is equal to* a.

165. *If it were proposed to multiply one cube root,* $\sqrt[3]{a}$, *by another,* $\sqrt[3]{b}$, *the product must be* $\sqrt[3]{ab}$; for we know that the cube root of a product ab is found by multiplying together the cube roots of the factors (156). Hence, also, *if we divide* $\sqrt[3]{a}$ *by* $\sqrt[3]{b}$, *the quotient will be* $\sqrt[3]{\frac{a}{b}}$.

166. We further perceive, that $2\sqrt[3]{a}$ *is equal to* $\sqrt[3]{8a}$, because 2 is equivalent to $\sqrt[3]{8}$; that $3\sqrt[3]{a}$ is equal to $\sqrt[3]{27a}$, and $b\sqrt[3]{a}$ is equal to $\sqrt[3]{abbb}$. So, reciprocally, if the number under the radical sign has a factor which is a cube, we may make it disappear by placing its cube root before the sign. For example, instead of $\sqrt[3]{64a}$ we may write $4\sqrt[3]{a}$; and $5\sqrt[3]{a}$ instead of $\sqrt[3]{125a}$. Hence $\sqrt[3]{16}$ *is equal to* $2\sqrt[3]{2}$, because 16 is equal to 8×2.

167. When a number proposed is negative, its cube root is not subject to the same difficulties that occurred in treating of square roots. For, since the cubes of negative numbers are negative, it follows that the cube roots of negative numbers are only negative. Thus, $\sqrt[3]{-8}$ is equal to -2, and $\sqrt[3]{-27}$ to -3. It follows also, that $\sqrt[3]{-12}$ is the same as $-\sqrt[3]{12}$, and that $\sqrt[3]{-a}$ may be expressed by $-\sqrt[3]{a}$. Whence we see, that the sign $-$, when it is found after the sign of the cube root, might also have been placed before it. We are not therefore here led to impossible, or imaginary numbers, as we were in considering the square roots of negative numbers.

CHAPTER XVI.

Of Powers in general.

168. THE *product, which we obtain by multiplying a number several times by itself, is called a power.* Thus, a square which arises from the multiplication of a number by itself, and a cube which we obtain by multiplying a number twice by itself, are powers. *We say also in the former case, that the number is raised to the second degree, or to the second power; and in the latter, that the number is raised to the third degree, or to the third power.*

169. We distinguish these powers from one another by the number of times that the given number has been used as a factor. For example, a square is called the second power, because a certain given number has been used twice as a factor; and if a number has been used thrice as a factor, we call the product the third power, which therefore means the same as the cube. Multiply a number by itself till you have used it four times as a factor, and you will have its fourth power, or what is commonly called the *bi-quadrate.* From what has been said it will be easy to understand what is meant by the fifth, sixth, seventh, &c., power of a number. I only add, that the names of these powers, after the fourth degree, cease to have any other but these numeral distinctions.

170. To illustrate this still further, we may observe, in the first place, that *the powers of* 1 *remain always the same;* because, whatever number of times we multiply 1 by itself, the product is found to be always 1. We shall therefore begin by representing the powers of 2 and of 3. They succeed in the following order:

Powers.	Of the number 2.	Of the number 3.
I.	2	3
II.	4	9
III.	8	27
IV.	16	81
V.	32	243
VI.	64	729
VII.	128	2187
VIII.	256	6561
IX.	512	19683
X.	1024	59049
XI.	2048	177147
XII.	4096	531441
XIII.	8192	1594323
XIV.	16384	4782969
XV.	32768	14348907
XVI.	65536	43046721
XVII.	131072	129140163
XVIII.	262144	387420489

But the powers of the number 10 are the most remarkable; for on these powers the system of our arithmetic is founded. A few of them arranged in order, and beginning with the first power, are as follows:

I.	II.	III.	IV.	V.	VI.
10.	100,	1000,	10000,	100000,	1000000, &c.

171. In order to illustrate this subject, and to consider it in a more general manner, we may observe, that the powers of any number, *a*, succeed each other in the following order.

I.	II.	III.	IV.	V.	VI.
a,	$a\,a$,	$a\,a\,a$,	$a\,a\,a\,a$,	$a\,a\,a\,a\,a$,	$a\,a\,a\,a\,a\,a$, &c.

But we soon feel the inconvenience attending this manner of writing powers, which consists in the necessity of repeating the same letter very often, to express high powers; and the reader also would have no less trouble, if he were obliged to count all the letters, to know what power is intended to be represented. The hundredth power, for example, could not be conveniently written in this manner; and it would be still more difficult to read it.

172. To avoid this inconvenience, a much more commodious method of expressing such powers has been devised, which from

its extensive use deserves to be carefully explained; *viz.* To express, for example, the hundredth power, we simply write the number 100 above the number whose hundredth power we would express, and a little towards the right-hand; thus a^{100} means a raised to 100, and *represents the hundreth power of* a. It must be observed, that *the name exponent is given to the number written above that whose power, or degree, it represents, and which in the present instance is* 100.

173. In the same manner, a^2 signifies a raised to 2, or the second power of a, which we represent sometimes also by aa, because both these expressions are written and understood with equal facility. But to express the cube, or the third power aaa, we write a^3 according to the rule, that we may occupy less room. So a^4 signifies the fourth, a^5 the fifth, and a^6 the sixth power of a.

174. In a word, all the powers of a will be represented by a, a^2, a^3. a^4. a^5, a^6, a^7, a^8, a^9, a^{10}, &c. Whence we see that in this manner we might very properly have written a^1 instead of a for the first term, to shew the order of the series more clearly. In fact a^1 *is no more than* a. *as this unit shews that the letter* a *is to be written only once.* Such a series of powers is called also a geometrical progression, because each term is greater by one than the preceding.

175. As in this series of powers each term is found by multiplying the preceding term by a, which increases the exponent by 1: so when any term is given, we may also find the preceding one, if we divide by a, because this diminishes the exponent by 1. This shews that the term which precedes the first term a^1 must necessarily be $\frac{a}{a}$, or 1; now, if we proceed according to the exponents, we immediately conclude, that the term which precedes the first must be a^0. Hence we deduce this remarkable property; that a^0 *is constantly equal to* 1, *however great or small the value of the number* a *may be,* and even when a is nothing: that is to say, a^0 is equal to 1.

176. We may continue our series of powers in a retrograde order, and that in two different ways; first, by dividing always by a, and secondly by diminishing the exponent by unity. And

it is evident that, whether we follow the one or the other, the terms are still perfectly equal. This decreasing series is represented, in both forms, in the following table, which must be read backwards, or from right to left.

	$\frac{1}{aaaaaa}$	$\frac{1}{aaaaa}$	$\frac{1}{aaaa}$	$\frac{1}{aaa}$	$\frac{1}{aa}$	$\frac{1}{a}$	1	a
1.	$\frac{1}{a^6}$	$\frac{1}{a^5}$	$\frac{1}{a^4}$	$\frac{1}{a^3}$	$\frac{1}{a^2}$	$\frac{1}{a^1}$		
2.	a^{-6}	a^{-5}	a^{-4}	a^{-3}	a^{-2}	a^{-1}	a^0	a^1

177. We are thus brought to understand the nature of powers, whose exponents are negative, and are enabled to assign the precise value of these powers. From what has been said, it appears that,

$$\left.\begin{array}{l} a^0 \\ a^{-1} \\ a^{-2} \\ a^{-3} \\ a^{-4} \end{array}\right\} \text{is equal to} \left\{\begin{array}{l} 1;\ \text{then} \\ \frac{1}{a}; \\ \frac{1}{aa};\ \text{or}\ \frac{1}{a^2}: \\ \frac{1}{a^3}; \\ \frac{1}{a^4},\ \text{\&c.} \end{array}\right.$$

178. It will be easy, from the foregoing notation, to find *the powers of a product,* a b. *They must evidently be* a b, *or* $a^1 b^1$, $a^2 b^2$, $a^3 b^3$, $a^4 b^4$, $a^5 b^5$, *&c. And the powers of fractions will be found in the same manner; for example those of* $\frac{a}{b}$ *are,*

$$\frac{a^1}{b^1}, \frac{a^2}{b^2}, \frac{a^3}{b^3}, \frac{a^4}{b^4}, \frac{a^5}{b^5}, \frac{a^6}{b^6}, \frac{a^7}{b^7}, \text{\&c.}$$

179. Lastly, we have to consider the powers of negative numbers. Suppose the given number to be $-a$; its powers will form the following series:

$$-a,\ +aa,\ -a^3,\ +a^4,\ -a^5,\ +a^6,\ \text{\&c.}$$

We may observe, that those powers only become negative whose exponents are odd numbers, and that, on the contrary, all the powers, which have an even number for the exponent, are positive. So that, the third, fifth, seventh, ninth, &c., powers have each the sign $-$; and the second, fourth, sixth, eighth, &c. powers are affected with the sign $+$.

CHAPTER XVII.

Of the calculation of Powers.

180. WE have nothing in particular to observe with regard to the addition and subtraction of powers; for we only represent these operations by means of the signs $+$ and $-$, when the powers are different. For example, $a^3 + a^2$ *is the sum of the second and third powers of* a; *and* $a^5 - a^4$ *is what remains when we subtract the fourth power of* a *from the fifth; and neither of these results can be abridged.* When we have powers of the same kind, or degree, it is evidently unnecessary to connect them by signs; $a^3 + a^3$ makes $2a^3$, &c.

181. But, in the multiplication of powers, several things require attention.

First, when it is required to multiply any power of a by a, we obtain the succeeding power, that is to say, the power whose exponent is greater by one unit. Thus a^2, multiplied by a, produces a^3; and a^3, multiplied by a, produces a^4. And, in the same manner, when it is required to multiply by a the powers of that number which have negative exponents, we must add 1 to the exponent. Thus, a^{-1} *multiplied by* a *produces* a^0 *or* 1; which is made more evident by considering that a^{-1} is equal to $\frac{1}{a}$, and that the product of $\frac{1}{a}$ by a being $\frac{a}{a}$, it is consequently equal to 1. Likewise a^{-2} multiplied by a produces a^{-1}, or $\frac{1}{a}$; and a^{-10}, multipled by a, gives a^{-9}, and so on.

182. Next, if it be required to multiply a power of a by aa, or the second power, I say that the exponent becomes greater by 2. Thus, the product of a^2 by a^2 is a^4; that of a^2 by a^3

a^5; that of a^4 by a^2 is a^6; and, more generally, a^n *multiplied by* a^2 *makes* a^{n+2}. *With regard to negative exponents, we shall have* a^1, *or* a, *for the product of* a^{-1} *by* a^2; for a^{-1} being equal to $\frac{1}{a}$, it is the same as if we had divided $a\,a$ by a; consequently the product required is $\frac{a\,a}{a}$, or a. So a^{-2}, *multiplied by* a^2 *produces* a^0, *or* 1; *and* a^{-3}, *multiplied by* a^2, *produces* a^{-1}.

183. It is no less evident that, to multiply any power of a by a^3, we must increase its exponent by three units; and that consequently the product of a^n by a^3 is a^{n+3}. And *whenever it is required to multiply together two powers of* a, *the product will be also a power of* a, *and a power whose exponent will be the sum of the exponents of the two given powers.* For example. a^4 multiplied by a^5 will make a^9, and a^{12} multiplied by a^7 will produce a^{19}, &c.

184. From these considerations we may easily determine the highest powers. To find, for instance, the twenty-fourth power of 2, I multiply the twelfth power by the twelfth power, because 2^{24} is equal to $2^{12} \times 2^{12}$. Now we have already seen that 2^{12} is 4096; I say therefore that the number 16777216, or the product of 4096 by 4096, expresses the power required, 2^{24}.

185. Let us proceed to division. We shall remark in the first place, that *to divide a power of* a *by* a, *we must subtract* 1 *from the exponent, or diminish it by unity.* Thus a^5, divided by a, gives a^4; a^0, or 1, divided by a, is equal to a^{-1} or $\frac{1}{a}$; a^{-3}, divided by a, gives a^{-4}.

186. If we have to divide a given power of a by a^2, we must diminish the exponent by 2; and if by a^3, we must subtract three units from the exponent of the power proposed. So, *in general, whatever power of* a *it is required to divide by another power of* a, *the rule is always to subtract the exponent of the second from the exponent of the first of these powers.* Thus a^{15}, divided by a^7, will give a^8; a^6 divided by a^7, will give a^{-1}; and a^{-3}, divided by a^4, will give a^{-7}.

187. From what has been said above, it is easy to understand the method of finding the powers of powers, this being done by multiplication. When we seek, for example, the square, or the second power of a^3, we find a^6; and in the same manner we

find a^{12} for the third power or the cube of a^4. *To obtain the square of a power, we have only to double its exponent; for its cube, we must triple the exponent; and so on.* The square of a^n is a^{2n}; the cube of a^n is a^{3n}; the seventh power of a^n is a^{7n}, &c.

188. The square of a^2, or the square of the square of a, being a^4, we see why the fourth power is called the *bi-quadrate*. The square of a^3 is a^6; the sixth power has therefore received the name of the *square-cubed*.

Lastly, the cube of a^3 being a^9, we call the ninth power the *cubo-cube*. No other denominations of this kind have been introduced for powers, and indeed the two last are very little used.

CHAPTER XVIII.

Of Roots with relation to Powers in general.

189. Since the square root of a given number is a number, whose square is equal to that given number; and since the cube root of a given number is a number, whose cube is equal to that given number; it follows that any number whatever being given, we may always indicate such roots of it, that their fourth, or their fifth, or any other power, may be equal to the given number. To distinguish these different kinds of roots better, we shall call the square root the *second root;* and the cube root the *third root;* because, according to this denomination, we may call the *fourth root,* that whose biquadrate is equal to a given number; and the *fifth root,* that whose fifth power is equal to a given number, &c.

190. As the square, or second root, is marked by the sign $\sqrt{}$, and the cubic or third root by the sign $\sqrt[3]{}$, so the fourth root is represented by the sign $\sqrt[4]{}$; the fifth root by the sign $\sqrt[5]{}$; and so on; it is evident that according to this mode of expression, the sign of the square root ought to be $\sqrt[2]{}$. But as of all roots this occurs most frequently, it has been agreed, for the sake of brevity, to omit the number 2 in the sign of this root. So that

when a radical sign has no number prefixed, this always shews that the square root is to be understood.

191. To explain this matter still further, we shall here exhibit the different roots of the number a, with their respective values:

$$\left.\begin{array}{l}\sqrt{a}\\ \sqrt[3]{a}\\ \sqrt[4]{a}\\ \sqrt[5]{a}\\ \sqrt[6]{a}\end{array}\right\}\text{is the}\left\{\begin{array}{c}\text{2d}\\ \text{3d}\\ \text{4th}\\ \text{5th}\\ \text{6th}\end{array}\right\}\text{root of}\left\{\begin{array}{l}a,\\ a,\\ a,\\ a,\\ a, \text{ and so on.}\end{array}\right.$$

So that conversely;

$$\left.\begin{array}{l}\text{The 2d}\\ \text{The 3d}\\ \text{The 4th}\\ \text{The 5th}\\ \text{The 6th}\end{array}\right\}\text{power of}\left\{\begin{array}{c}\sqrt{a}\\ \sqrt[3]{a}\\ \sqrt[4]{a}\\ \sqrt[5]{a}\\ \sqrt[6]{a}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}a,\\ a,\\ a,\\ a,\\ a,\end{array}\right.$$

and so on.

192. Whether the number a therefore be great or small, we know what value to affix to all these roots of different degrees.

It must be remarked also, that if we substitute unity for a, all those roots remain constantly 1; because all the powers of 1 have unity for their value. If the number a be greater than 1, all its roots will also exceed unity. Lastly, if that number be less than 1, all its roots will also be less than unity.

193. When the number a is positive, we know from what was before said of the square and cube roots, that all the other roots may also be determined, and will be real and possible numbers.

But if the number a is negative, its second, fourth, sixth, and all the even roots, become impossible, or imaginary numbers; because *all the even powers, whether of positive, or of negative numbers, are affected with the sign +. Whereas the third, fifth, seventh, and all odd roots, become negative,* but rational; because the odd powers of negative numbers, are also negative.

194. We have here also an inexhaustible scource of new kinds of surd, or irrational quantities; for whenever the number a is not actually such a power, as some one of the foregoing indices represents, or seems to require, it is impossible to express that root either in whole numbers or in fractions; and consequently it must be classed among the numbers which are called irrational.

CHAPTER XIX.

Of the Method of representing Irrational Numbers by Fractional Exponents.

195. We have shewn in the preceding chapter, that the square of any power is found by doubling the exponent of that power, and that in general the square, or the second power of a^n, is a^{2n}. The converse follows, namely, that *the square root of the power* a^{2n} is a^n, and that *it is found by taking half the exponent of that power, or dividing it by* 2.

196. Thus the square root of a^2 is a^1; that of a^4 is a^2; that of a^6 is a^3; and so on. And as this is general, the square root of a^3 must necessarily be $a^{\frac{3}{2}}$ and that of a^5 $a^{\frac{5}{2}}$. Consequently we shall have in the same manner $a^{\frac{1}{2}}$ for the square root of a^1; whence we see that $a^{\frac{1}{2}}$ *is equal to* $\sqrt{a}$; and this new method of representing the square root demands particular attention.

197. We have also shewn that, to find the cube of a power as a^n, we must multiply its exponent by 3, and that consequently the cube is a^{3n}.

So conversely, when it is required to find the third or cube root of the power a^{3n}, we have only to divide the exponent by 3, and may with certainty conclude, that the root required is a^n. Consequently a^1, or a, is the cube root of a^3; a^2 is that of a^6; a^3 is that of a^9; and so on.

198. There is nothing to prevent us from applying the same reasoning to those cases in which the exponent is not divisible

by 3, and concluding that the cube root of a^2 is $a^{\frac{2}{3}}$, and that the cube root of a^4 is $a^{\frac{4}{3}}$, or $a^{1\frac{1}{3}}$. Consequently the third, or cube root of a also, or a^1 must be $a^{\frac{1}{3}}$. Whence it appears that $a^{\frac{1}{3}}$ *is equal to* $\sqrt[3]{a}$.

199. It is the same with roots of a higher degree. The fourth root of a will be $a^{\frac{1}{4}}$, which expression has the same value as $\sqrt[4]{a}$. The fifth root of a will be $a^{\frac{1}{5}}$, which is consequently equivalent to $\sqrt[5]{a}$; and the same observation may be extended to all roots of a higher degree.

200. We might therefore entirely reject the radical signs at present made use of, and employ in their stead the fractional exponents which we have explained; however, as we have been long accustomed to those signs, and meet with them in all books of algebra, it would be wrong to banish them entirely. But there is sufficient reason also to employ, as is now frequently done, the other method of notation, because it manifestly corresponds with what is to be represented. In fact, we see immediately that $a^{\frac{1}{2}}$ is the square root of a, because we know that the square of $a^{\frac{1}{2}}$, that is to say, $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{2}}$, is equal to a^1 or a.

201. What has now been said is sufficient to shew how we are to understand all other fractional exponents that may occur. If we have, for example, $a^{\frac{4}{3}}$, this means that we must first take the fourth power of a, and then extract its cube or third root; so that $a^{\frac{4}{3}}$ *is the same as the common expression*, $\sqrt[3]{a^4}$. To find the value of $a^{\frac{3}{4}}$, we must first take the cube, or the third power of a, which is a^3, and then extract the fourth root of that power; so that $a^{\frac{3}{4}}$ is the same as $\sqrt[4]{a^3}$. Also $a^{\frac{4}{5}}$ is equal to $\sqrt[5]{a^4}$, &c.

202. When the fraction which represents the exponent exceeds unity, we may express the value of the given quantity in another way. Suppose it to be $a^{\frac{5}{2}}$; this quantity is equivalent to $a^{2\frac{1}{2}}$, which is the product of a^2 by $a^{\frac{1}{2}}$. Now $a^{\frac{1}{2}}$ being

equal to $\sqrt{a}$, it is evident that $a^{\frac{5}{2}}$ is equal to $a^2\sqrt{a}$. So $a^{\frac{10}{3}}$, or $a^{3\frac{1}{3}}$ is equal to $a^3\sqrt[3]{a}$; and $a^{\frac{15}{4}}$, that is $a^{3\frac{3}{4}}$, expresses $a^3\sqrt[4]{a^3}$. These examples are sufficient to illustrate the great utility of fractional exponents.

203. Their use extends also to fractional numbers: let there be given $\frac{1}{\sqrt{a}}$, we know that this quantity is equal to $\frac{1}{a^{\frac{1}{2}}}$; now we have seen already that a fraction of the form $\frac{1}{a^n}$ may be expressed by a^{-n}; so instead of $\frac{1}{\sqrt{a}}$ we may use the expression $a^{-\frac{1}{2}}$. In the same manner, $\frac{1}{\sqrt[3]{a}}$ *is equal to* $a^{-\frac{1}{3}}$. Again, let the quantity $\frac{a^2}{\sqrt[4]{a^3}}$ be proposed; let it be transformed into this, $\frac{a^2}{a^{\frac{3}{4}}}$, which is the product of a^2 by $a^{-\frac{3}{4}}$; now this product is equivalent to $a^{\frac{5}{4}}$, or to $a^{1\frac{1}{4}}$, or lastly to $a\sqrt[4]{a}$. Practice will render similar reductions easy.

204. We shall observe, in the last place, that each root may be represented in a variety of ways. For $\sqrt{a}$ being the same as $a^{\frac{1}{2}}$, and $\frac{1}{2}$ being transformable into all these fractions, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, &c., it is evident that $\sqrt{a}$ *is equal to* $\sqrt[4]{a^2}$, and to $\sqrt[6]{a^3}$ and to $\sqrt[8]{a^4}$, and so on. In the same manner $\sqrt[3]{a}$, which is equal to $a^{\frac{1}{3}}$, will be equal to $\sqrt[6]{a^2}$, and to $\sqrt[9]{a^3}$, and to $\sqrt[12]{a^4}$. And we see also, that the number a, or a^1, might be represented by the following radical expressions:

$$\sqrt[2]{a^2},\ \sqrt[3]{a^3},\ \sqrt[4]{a^4},\ \sqrt[5]{a^5},\ \&c.$$

205. This property is of great use in multiplication and division: for if we have, for example, to multiply $\sqrt[2]{a}$ by $\sqrt[3]{a}$, we write $\sqrt[6]{a^3}$ for $\sqrt[2]{a}$, and $\sqrt[6]{a^2}$ instead of $\sqrt[3]{a}$; in this manner we obtain the same radical sign for both, and the multiplication being now performed, gives the product $\sqrt[6]{a^5}$. The same result is deduced from $a^{\frac{1}{2}+\frac{1}{3}}$, the product of $a^{\frac{1}{2}}$ multi-

plied by $a^{\frac{1}{3}}$; for $\frac{1}{2}+\frac{1}{3}$ is $\frac{5}{6}$, and consequently the product required is $a^{\frac{5}{6}}$ or $\sqrt[6]{a^5}$.

If it were required to divide $\sqrt[2]{a}$, *or* $a^{\frac{1}{2}}$, *by* $\sqrt[3]{a}$ *or* $a^{\frac{1}{3}}$, *we should have for the quotient* $a^{\frac{1}{2}-\frac{1}{3}}$, *or* $a^{\frac{3}{6}-\frac{2}{6}}$, *that is say,* $a^{\frac{1}{6}}$ *or* $\sqrt[6]{a}$.

CHAPTER XX.

Of the different methods of calculation, and of their mutual connexion.

206. Hitherto we have only explained the different methods of calculation: addition, subtraction, multiplication, and division; the involution of powers, and the extraction of roots. It will not be improper therefore, in this place, to trace back the origin of these different methods, and to explain the connexion which subsists among them; in order that we may satisfy ourselves whether it be possible or not for other operations of the same kind to exist. This inquiry will throw new light on the subjects which we have considered.

In prosecuting this design, we shall make use of a new character, which may be employed instead of the expression that has been so often repeated, *is equal to;* this sign is $=$, and is read *is equal to.* Thus, when I write $a=b$, this means that a is
3×5 equal to b: so, for example $\cancel{3+5}=15$.

207. The first mode of calculation, which presents itself to the mind, is undoubtedly addition, by which we add two numbers together and find their sum. Let a and b then be the two given numbers, and let their sum be expressed by the letter c, we shall have $a+b=c$. So that when we know the two numbers a and b, addition teaches us to find the number c.

208. Preserving this comparison $a+b=c$, let us reverse the question by asking, how we are to find the number b, when we know the numbers a and c.

It is required therefore to know what number must be added to a, in order that the sum may be the number c. Suppose, for example, $a=3$ and $c=8$; so that we must have $3+b=8$;

b will evidently be found by subtracting 3 from 8. So, in general, to find b, we must subtract a from c, whence arises $b = c - a$; for by adding a to both sides again, we have $b + a = c - a + a$, that is to say $= c$, as we supposed.

Such then is the origin of subtraction.

209. Subtraction therefore takes place, when we invert the question which gives rise to addition. Now the number which it is required to subtract may happen to be greater than that from which it is to be subtracted; as, for example, if it were required to subtract 9 from 5: this instance therefore furnishes us with the idea of a new kind of numbers, which we call negative numbers, because $5 - 9 = -4$.

210. When several numbers are to be added together which are all equal, their sum is found by multiplication, and is called a product. Thus $a b$ means the product arising from the multiplication of a by b, or from the addition of a number a to itself b number of times. If we represent this product by the letter c, we shall have $a b = c$; and multiplication teaches us how to determine the number c, when the numbers a and b are known.

211. Let us now propose the following question: the numbers a and c being known, to find the number b. Suppose, for example, $a = 3$ and $c = 15$, so that $3 b = 15$, we ask by what number 3 must be multiplied, in order that the product may be 15: for the question proposed is reduced to this. Now this is division: the number required is found by dividing 15 by 3; and therefore, in general, the number b is found by dividing c by a; from which results the equation $b = \frac{c}{a}$.

212. Now, as it frequently happens that the number c cannot be really divided by the number a, while the letter b must however have a determinate value, another new kind of numbers presents itself; these are fractions. For example, supposing $a = 4$, $c = 3$, so that $4 b = 3$, it is evident that b cannot be an integer, but a fraction, and that we shall have $b = \frac{3}{4}$.

213. We have seen that multiplication arises from addition, that is to say, from the addition of several equal quantities. If we now proceed further, we shall perceive that from the multiplication of several equal quantities to-

gether powers are derived. Those powers are represented in a general manner by the expression a^b, which signifies that the number a must be multiplied as many times by itself, as is denoted by the number b. And we know from what has been already said, that in the present instance a is called the root, b the exponent, and a^b the power.

214. Further, if we represent this power also by the letter c, we have $a^b = c$, an equation in which three letters a, b, c, are found. Now we have shewn in treating of powers, how to find the power itself, that is, the letter c, when a root a and its exponent b are given. Suppose, for example, $a = 5$, and $b = 3$, so that $c = 5^3$; it is evident that we must take the third power of 5, which is 125, and that thus $c = 125$.

215. We have seen how to determine the power c, by means of the root a and the exponent b; but if we wish to reverse the question, we shall find that this may be done in two ways, and that there are two different cases to be considered: for if two of these three numbers a, b, c, were given, and it were required to find the third, we should immediately perceive that this question admits of three different suppositions, and consequently three solutions. We have considered the case in which a and b were the numbers given, we may therefore suppose further that c and a, or c and b are known, and that it is required to determine the third letter. Let us point out therefore, before we proceed any further, a very essential distinction between involution and the two operations which lead to it. When in addition we reversed the question, it could be done only in one way; it was a matter of indifference whether we took c and a, or c and b, for the given numbers, because we might indifferently write $a + b$, or $b + a$. It was the same with multiplication; we could at pleasure take the letters a and b for each other, the equation $ab = c$ being exactly the same as $ba = c$.

In the calculation of powers, on the contrary, the same thing does not take place, and we can by no means write b^a instead of a^b. A single example will be sufficient to illustrate this: let $a = 5$, and $b = 3$; we have $a^b = 5^3 = 125$. But $b^a = 3^5 = 243$: two very different results.

SECTION II.

OF THE DIFFERENT METHODS OF CALCULATION APPLIED TO COMPOUND QUANTITIES.

CHAPTER I.

Of the Addition of Compound Quantities.

ARTICLE 216.

When two or more expressions, consisting of several terms, are to be added together, the operation is frequently represented merely by signs, placing each expression between two parentheses, and connecting it with the rest by means of the sign $+$. If it be required, for example, to add the expressions $a+b+c$ and $d+e+f$, we represent the sum thus:

$$(a+b+c)+(d+e+f).$$

217. It is evident that this is not to perform addition, but only to represent it. We see at the same time, however, that in order to perform it actually, we have only to leave out the parentheses; for as the number $d+e+f$ is to be added to the other, we know that this is done by joining to it first $+d$, then $+e$, and then $+f$; which therefore gives the sum

$$a+b+c+d+e+f.$$

The same method is to be observed, if any of the terms are affected with the sign $-$; they must be joined in the same way, by means of their proper sign.

218. To make this more evident, we shall consider an example in pure numbers. It is proposed to add the expression $15-6$ to $12-8$. If begin by adding 15, we shall have $12-8+15$; now this was adding too much, since we had only to add $15-6$, and it is evident that 6 is the number which we have added too much. Let us therefore take this 6 away by writing it with the negative sign, and we shall have the true sum, $12-8+15-6$,

which shews that the *sums are found by writing all the terms, each with its proper sign.*

219. If it were required therefore to add the expression $d-e-f$ to $a-b+c$, we should express the sum thus:

$$a-b+c+d-e-f,$$

remarking however that it is of no consequence in what order we write these terms. Their place may be changed at pleasure, provided their signs be preserved. This sum might, for example, be written thus:

$$c-e+a-f+d-b.$$

220. It frequently happens, that the sums represented in this manner may be considerably abridged, as when two or more terms destroy each other; for example, if we find in the same sum the terms $+a-a$, or $3a-4a+a$: or when two or more terms may be reduced to one. Examples of this second reduction:

$$3a+2a=5a;\ 7b-3b=+4b;$$
$$-6c+10c=+4c;$$
$$5a-8a=-3a;\ -7b+b=-6b;$$
$$-3c-4c=-7c;$$
$$2a-5a+a=-2a;\ -3b-5b+2b=-6b.$$

Whenever two or more terms, therefore, *are entirely the same with regard to letters,* their sum may be abridged: but those cases must not be confounded with such as these, $2aa+3a$, or $2b^3-b^4$, which admit of no abridgment.

221. Let us consider some more examples of reduction; the following will lead us immediately to an important truth. Suppose it were required to add together the expressions $a+b$ and $a-b$; our rule gives $a+b+a-b$; now $a+a=2a$ and $b-b=0$; the sum then is $2a$: consequently if we add together the sum of two numbers $(a+b)$ and their difference $(a-b,)$ we obtain the double of the greater of those two numbers.

Further examples:

$$\begin{array}{r|l} 3a-2b-c & a^3-2aab+2abb \\ 5b-6c+a & -aab+2abb-b^3 \\ \hline 4a+3b-7c & a^3-3aab+4abb-b^3. \end{array}$$

CHAPTER II.

Of the Subtraction of Compound Quantities.

222. If we wish merely to represent subtraction, we inclose each expression within two parentheses, connecting, by the sign $-$, the expression to be subtracted with that from which it is to be taken.

When we subtract, for example, the expression $d-e+f$ from the expression $a-b+c$, we write the remainder thus:

$$(a-b+c)-(d-e+f);$$

and this method of representing it sufficiently shews, which of the two expressions is to be subtracted from the other.

223. But if we wish to perform the subtraction, we must observe, first, that when we subtract a positive quantity $+b$ from another quantity a, we obtain $a-b$: and secondly, when we subtract a negative quantity $-b$ from a, we obtain $a+b$; because to free a person from a debt is the same as to give him something.

224. Suppose, now, it were required to subtract the expression $b-d$ from the expression $a-c$, we first take away b; which gives $a-c-b$. Now this is taking too much away by the quantity d, since we had to subtract only $b-d$; we must therefore restore the value of d, and we shall then have

$$a-c-b+d;$$

whence it is evident, that *the terms of the expression to be subtracted must have their signs changed, and be joined, with the contrary signs, to the terms of the other expression.*

225. It is easy, therefore, by means of this rule, to perform subtraction, since we have only to write the expression from which we are to subtract, such as it is, and join the other to it without any change beside that of the signs. Thus, in the first example, where it was required to subtract the expression $d-e+f$ from $a-b+c$, we obtain $a-b+c-d+e-f$.

An example in numbers will render this still more clear. If we subtract $6-2+4$ from $9-3+2$, we evidently obtain

$$9-3+2-6+2-4;$$

for $9-3+2=8$; also, $6-2+4=8$; now $8-8=0$.

226. Subtraction being therefore subject to no difficulty, we have only to remark, that, if there are found in the remainder two, or more terms which are entirely similar with regard to the letters, that remainder may be reduced to an abridged form, by the same rules which we have given in addition.

227. Suppose we have to subtract from $a+b$, or from the sum of two quantities, their difference $a-b$, we shall then have $a+b-a+b$; now $a-a=0$, and $b+b=2b$; the remainder sought is therefore $2b$, that is to say, the double of the less of the two quantities.

228. The following examples will supply the place of further illustrations.

$aa+ab+bb$	$3a-4b+5c$	$a^3+3aab+3abb+b^3$	$\sqrt{a}+2\sqrt{b}$
$bb+ab-aa$	$2b+4c-6a$	$a^3-3aab+3abb-b^3$	$\sqrt{a}-3\sqrt{b}$
$2aa.$	$9a-6b+c.$	$6aab+2b^3.$	$+5\sqrt{b}.$

CHAPTER III.

Of the Multiplication of Compound Quantities.

229. When it is only required to represent multiplication, we put each of the expressions, that are to be multiplied together, within two parentheses, and join them to each other, sometimes without any sign, and sometimes placing the sign $\times$ between them. For example, to represent the product of the two expressions $a-b+c$ and $d-e+f$, when multiplied together, we write.

$$(a-b+c)\times(d-e+f.)$$

This method of expressing products is much used, because it immediately shews the factors of which they are composed.

230. But to shew how multiplication is to be actually performed, we may remark, in the first place, that in order to multiply, for example, a quantity, such as $a-b+c$, by 2, each term of it is separately multiplied by that number; so that the product is

$$2a-2b+2c.$$

Now the same thing takes place with regard to all other numbers. If d were the number, by which it is required to multiply the same expression, we should obtain

$$a\,d - b\,d + c\,d.$$

231. We supposed d to be a positive number; but if the factor were a negative number, as $-e$, the rule heretofore given must be applied; namely, that *two contrary signs, multiplied together, produce* $-$, *and that two like signs give* $+$.

We shall accordingly have

$$-a\,e + b\,e - c\,e.$$

232. To shew how a quantity, A, is to be multiplied by a compound quantity, $d - e$; let us first consider an example in common numbers, supposing that A is to be multiplied by $7 - 3$. Now it is evident, that we are here required to take the quadruple of A; for if we first take A seven times, it will then be necessary to subtract $3\,A$ from that product.

In general, therefore, if it be required to multiply by $d - e$, we multiply the quantity A first by d and then by e, and subtract this last product from the first; whence results $d\,A - e\,A$.

Suppose now $A = a - b$, and that this is the quantity to be multiplied by $d - e$; we shall have

$$\begin{array}{l} d\,A = a\,d - b\,d \\ e\,A = a\,e - b\,e \\ \hline \end{array}$$

whence the product required $= a\,d - b\,d - a\,e + b\,e$.

233. Since we know therefore the product $(a - b) \times (d - e,)$ and cannot doubt of its accuracy, we shall exhibit the same example of multiplication under the following form:

$$\begin{array}{c} a - b \\ d - e \\ \hline a\,d - b\,d - a\,e + b\,e. \end{array}$$

This shews, that we must *multiply each term of the upper expression by each term of the lower*, and that, with regard to the signs, we must strictly observe the rule before given; a rule which this would completely confirm, if it admitted of the least doubt.

234. It will be easy, according to this rule, to perform the following example, which is, to multiply $a+b$ by $a-b$:

$$\begin{array}{l} a+b \\ a-b \\ \hline aa+ab \\ \quad -ab-bb \\ \hline \text{Product } aa-bb. \end{array}$$

235. Now we may substitute, for a and b, any determinate numbers; so that the above example will furnish the following theorem; viz. *The product of the sum of two numbers, multiplied by their difference, is equal to the difference of the squares of those numbers.* This theorem may be expressed thus:

$$(a+b)\times(a-b)=aa-bb.$$

And from this, another theorem may be derived; namely, *The difference of two square numbers is always a product, and divisible both by the sum and by the difference of the roots of those two squares.*

236. Let us now perform some other examples:

$$\begin{array}{l} \text{I.)}\ 2a-3 \\ \quad a+2 \\ \hline 2aa-3a \\ \quad +4a-6 \\ \hline 2aa+a-6. \end{array}$$

$$\begin{array}{l} \text{II.)}\ 4aa-6a+9 \\ \quad 2a+3 \\ \hline 8a^3-12aa+18a \\ \quad +12aa-18a+27 \\ \hline 8a^3+27 \end{array}$$

$$
\text{III.)}\quad
\begin{array}{l}
3aa - 2ab - bb \\
\;\; 2a - 4b \\
\hline
6a^3 - 4aab - 2abb \\
\qquad - 12aab + 8abb + 4b^3 \\
\hline
6a^3 - 16aab + 6abb + 4b^3
\end{array}
$$

$$
\text{IV.)}\quad
\begin{array}{l}
aa + 2ab + 2bb \\
aa - 2ab + 2bb \\
\hline
a^4 + 2a^3b + 2aabb \\
\quad - 2a^3b - 4aabb - 4ab^3 \\
\qquad\qquad + 2aabb + 4ab^3 + 4b^4 \\
\hline
\qquad a^4 + 4b^4.
\end{array}
$$

$$
\text{V.)}\quad
\begin{array}{l}
2aa - 3ab - 4bb \\
3aa - 2ab + bb \\
\hline
6a^4 - 9a^3b - 12aabb \\
\qquad - 4a^3b + 6aabb + 8ab^3 \\
\qquad\qquad + 2aabb - 3ab^3 - 4b^4 \\
\hline
6a^4 - 13a^3b - 4aabb + 5ab^3 - 4b^4
\end{array}
$$

$$
\text{VI.)}\quad
\begin{array}{l}
aa + bb + cc - ab - ac - bc \\
\;\; a + b + c \\
\hline
a^3 + abb + acc - aab - aac - abc \\
\qquad aab + b^3 + bcc - abb - abc - bbc \\
\qquad\qquad aac + bbc + c^3 - abc - acc - bcc \\
\hline
\qquad\quad a^3 - 3abc + b^3 + c^3.
\end{array}
$$

237. *When we have more than two quantities to multiply together, it will easily be understood that, after having multiplied two of them together, we must then multiply that product by one of those which remain, and so on. It is indifferent what order is observed in these multiplications.*

Let it be proposed, for example, to find the value, or product, of the four following factors, *viz.*

$$\overset{\text{I.}}{(a+b)}\ \overset{\text{II.}}{(aa+ab+bb)}\ \overset{\text{III.}}{(a-b)}\ \overset{\text{IV.}}{(aa-ab+bb)}.$$

We will first multiply the factors I. and II.

$$\begin{array}{rl}
\text{II.} & aa+ab+bb \\
\text{I.} & a+b \\ \hline
 & a^3+aab+abb \\
 & \quad +aab+abb+b^3 \\ \hline
\text{I. II.} & a^3+2aab+2abb+b^3.
\end{array}$$

Next let us multiply the factors III. and IV.

$$\begin{array}{rl}
\text{IV.} & aa-ab+bb \\
\text{III.} & a-b \\ \hline
 & a^3-aab+abb \\
 & \quad -aab+abb-b^3 \\ \hline
\text{III. IV.} & a^3-2aab+2abb-b^3.
\end{array}$$

It remains now to multiply the first product I. II. by this second product III. IV. :

$$\begin{array}{llllllll}
a^3 & +2aab & +2abb & +b^3 & \text{I. II.} \\
a^3 & -2aab & +2abb & -b^3 & \text{III. IV.} \\ \hline
a^6 & +2a^5b & +2a^4bb & +a^3b^3 \\
 & -2a^5b & -4a^4bb & -4a^3b^3 & -2aab^4 \\
 & & 2a^4bb & +4a^3b^3 & +4aab^4 & +2ab^5 \\
 & & & -a^3b^3 & -2aab^4 & -2ab^5 & -b^6 \\ \hline
\end{array}$$

$$a^6-b^6.$$

And this is the product required.

238. Let us resume the same example, but change the order of it, first multiplying the factors I. and III. and then II. and IV. together.

$$\begin{array}{rl}
\text{I.} & a+b \\
\text{III.} & a-b \\ \hline
 & aa+ab \\
 & \quad -ab-bb \\ \hline
\text{I. III.} & =aa-bb.
\end{array}$$

$$
\begin{array}{rl}
\text{II.} & aa+ab+bb \\
\text{IV.} & aa-ab+bb \\
\hline
 & a^4+a^3b+aabb \\
 & \quad -a^3b-aabb-ab^3 \\
 & \qquad\qquad aabb+ab^3+b^4. \\
\hline
\end{array}
$$

$$\text{II. IV.} = a^4+aabb+b^4.$$

Then multiplying the two products I. III. and II. IV.

$$
\begin{array}{rl}
\text{II. IV.} = & a^4+aabb+b^4 \\
\text{I. III.} = & aa-bb \\
\hline
 & a^6+a^4bb+aab^4 \\
 & \quad -a^4b \quad -aab^4-b^6 \\
\hline
\end{array}
$$

we have a^6-b^6,

which is the product required.

239. We shall perform this calculation in a still different manner, first multiplying the Ist. factor by the IVth. and next the II^{d}. by the III^{d}.

$$
\begin{array}{rl}
\text{IV.} & aa-ab+bb \\
\text{I.} & a+b \\
\hline
 & a^3-aab+abb \\
 & \qquad abb-abb+b^3 \\
\hline
\end{array}
$$

$$\text{I. IV.} = a^3+b^3.$$

$$
\begin{array}{rl}
\text{II.} & aa+ab+bb \\
\text{III.} & a-b \\
\hline
 & a^3+aab+abb \\
 & \quad -aab-abb-b^3 \\
\hline
\end{array}
$$

$$\text{II. III.} = a^3-b^3.$$

It remains to multiply the product I. IV. and II. III.

$$
\begin{array}{rl}
\text{I. IV.} = & a^3+b^3 \\
\text{II. III.} = & a^3-b^3 \\
\hline
 & a^6+a^3b^3 \\
 & \quad -a^3b^3-b^6 \\
\hline
\end{array}
$$

and we still obtain a^6-b^6, as before.

240. It will be proper to illustrate this example by a numerical application. Let us make $a=3$ and $b=2$, we shall have $a+b=5$ and $a-b=1$; further, $aa=9$, $ab=6$, $bb=4$. Therefore $aa+ab+bb=19$, and $aa-ab+bb=7$. So that the product required is that of $5\times 19\times 1\times 7$, which is 665.

Now $a^6=729$, and $b^6=64$, consequently the product required is $a^6-b^6=665$, as we have already seen.

CHAPTER IV.

Of the Divison of Compound Quantities.

141. When we wish simply to represent division, we make use of the usual mark of fractions, which is, to write the denominator under the numerator, separating them by a line; or to inclose each quantity between a parenthesis, placing two points between the divisor and dividend. If it were required, for example to divide $a+b$ by $c+d$, we should represent the quotient thus $\frac{a+b}{c+d}$, according to the former method; and thus, $(a+b):(c+d)$ according to the latter. Each expression is read $a+b$ divided by $c+d$.

242. *When it is required to divide a compound quantity by a simple one, we divide each term separately.* For example;

$$6a-8b+4c, \text{ divided by } 2, \text{ gives } 3a-4b+2c;$$

$$\text{and } (aa-2ab):(a)=a-2b.$$

In the same manner

$$(a^3-2aab+3aab):(a)=aa-2ab+3ab;$$

$$(4aab-6aac+8abc):(2a)=2ab-3ac+4bc;$$

$$(9aabc-12abbc+15abcc):(3abc)=3a-4b+5c, \text{ \&c.}$$

243. If it should happen that a term of the dividend is not divisible by the divisor, the quotient is represented by a fraction, as in the division of $a+b$ by a, which gives $1+\frac{b}{a}$. Likewise,

$$(aa-ab+bb):(aa)=1-\frac{b}{a}+\frac{bb}{aa}.$$

For the same reason, if we divide $2a+b$ by 2, we obtain $a+\frac{b}{2}$; and here it may be remarked, that we may write $\frac{1}{2}b$, instead of $\frac{b}{2}$, because $\frac{1}{2}$ times b is equal to $\frac{b}{2}$. In the same manner $\frac{b}{3}$ is the same as $\frac{1}{3}b$, and $\frac{2b}{3}$ the same as $\frac{2}{3}b$, &c.

244. But when the divisor is itself a compound quantity, division becomes more difficult. Sometimes it occurs where we least expect it; but when it cannot be performed, we must content ourselves with representing the quotient by a fraction, in the manner that we have already described. Let us begin by considering some cases, in which actual division succeeds.

245. Suppose it were required to divide the dividend $ac - bc$ by the divisor $a - b$, *the quotient must* then *be such as, when multiplied by the divisor* a — b, *will produce the dividend* a c — b c. Now it is evident, that this quotient must include c, since without it we could not obtain ac. In order, therefore, to try whether c is the whole quotient, we have only to multiply it by the divisor, and see if that multiplication produces the whole dividend, or only part of it. In the present case, if we multiply $a - b$ by c, we have $ac - bc$, which is exactly the dividend; so that c is the whole quotient. It is no less evident, that

$$(aa + ab) : (a + b) = a; \quad (3aa - 2ab) : (3a - 2b) = a;$$
$$(6aa - 9ab) : (2a - 3b) = 3a, \text{ \&c.}$$

246. *We cannot fail, in this way, to find a part of the quotient; if, therefore, what we have found, when multiplied by the divisor, does not yet exhaust the dividend, we have only to divide the remainder again by the divisor, in order to obtain a second part of the quotient; and to continue the same method, until we have found the whole quotient.*

Let us, as an example, divide $aa + 3ab + 2bb$ by $a + b$; it is evident, in the first place, that the quotient will include the term a, since otherwise we should not obtain aa. Now, from the multiplication of the divisor $a + b$ by a, arises $aa + ab$; which quantity being subtracted from the dividend, leaves a remainder $2ab + 2bb$. This remainder must also be divided by $a + b$; and it is evident that the quotient of this division must contain the term $2b$. Now $2b$, multiplied by $a + b$, produces exactly $2ab + 2bb$; consequently $a + 2b$ is the quotient required; which, multiplied by the divisor $a+b$, ought to produce the dividend $aa+3ab+2bb$. See the whole operation:

$$\begin{array}{rl} a+b) & aa+3ab+2bb\,(a+2b \\ & aa+ab \\ \hline & \quad 2ab+2bb \\ & \quad 2ab+2bb \\ \hline & \quad\quad 0. \end{array}$$

247. *This operation will be facilitated if we choose one of the terms of the divisor to be written first, and then, in arranging the terms of the dividend, begin with the highest powers of that first term of the divisor.* This term in the preceding example was a; the following examples will render the operation more clear.

$$\begin{array}{rl} a-b) & a^3-3aab+3abb-b^3\,(aa-2ab+bb \\ & a^3-aab \\ \hline & -2aab+3abb \\ & -2aab+2abb \\ \hline & \quad\quad abb-b^3 \\ & \quad\quad abb-b^3 \\ \hline & \quad\quad\quad 0. \end{array}$$

$$\begin{array}{rl} a+b) & aa-bb\,(a-b \\ & aa+ab \\ \hline & \quad -ab-bb \\ & \quad -ab-bb \\ \hline & \quad\quad 0. \end{array}$$

$$\begin{array}{rl} 3a-2b) & 18aa-8bb\,(6a+4b \\ & 18aa-12ab \\ \hline & \quad\quad 12ab-8bb \\ & \quad\quad 12ab-8bb \\ \hline & \quad\quad\quad 0. \end{array}$$

$$a+b)\ a^3+b^3\ (aa-ab+bb$$
$$a^3+aab$$
$$-aab+b^3$$
$$-aab-abb$$
$$abb+b^3$$
$$abb+b^3$$
$$0.$$

$$2a-b)\ 8a^3-b^3\ (4aa+2ab+bb$$
$$8a^3-4aab$$
$$4aab-b^3$$
$$4aab-2abb$$
$$2abb-b^3$$
$$2abb-b^3$$
$$0.$$

$$aa-2ab+bb)\ a^4-4a^3b+6aabb-4ab^3+b^4$$
$$aa-2ab+bb)\ a^4-2a^3b+aabb$$
$$-2a^3b+5aabb-4ab^3$$
$$-2a^3b+4aabb-2ab^3$$
$$aabb-2ab^3+b^4$$
$$aabb-2ab^3+b^4$$
$$0.$$

$$aa-2ab+4bb)\ a^4+4aabb+16b^4(\ aa+2ab+4bb$$
$$a^4-2a^3b+4aabb$$
$$2a^3b+16b^4$$
$$2a^3b-4aabb+8ab^3$$
$$4aabb-8ab^3+16b^4$$
$$4aabb-8ab^3+16b^4$$
$$0.$$

$$
\begin{array}{l}
aa - 2ab + 2bb)\ a^4 + 4b^4\ (aa + 2ab + 2bb \\
\qquad\qquad\qquad a^4 - 2a^3b + 2aabb \\
\hline
\qquad\qquad\qquad\qquad 2a^3b - 2aabb + 4b^4 \\
\qquad\qquad\qquad\qquad 2a^3b - 4aabb + 4ab^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad 2aabb - 4ab^3 + 4b^4 \\
\qquad\qquad\qquad\qquad\qquad 2aabb - 4ab^3 + 4b^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad 0.
\end{array}
$$

$$
\begin{array}{l}
\qquad 1 - 2x + xx)\ 1 - 5x + 10xx - 10x^3 + 5x^4 - x^5 \\
1 - 3x + 3xx - x^3)\ 1 - 2x + xx \\
\hline
\qquad\qquad\qquad - 3x + 9xx - 10x^3 \\
\qquad\qquad\qquad - 3x + 6xx - 3x^3 \\
\hline
\qquad\qquad\qquad\qquad 3xx - 7x^3 + 5x^4 \\
\qquad\qquad\qquad\qquad 3xx - 6x^3 + 3x^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad - x^3 + 2x^4 - x^5 \\
\qquad\qquad\qquad\qquad\qquad - x^3 + 2x^4 - x^5 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad 0.
\end{array}
$$

CHAPTER V.

Of the Resolution of Fractions into infinite series.

248. When the dividend is not divisible by the divisor, the quotient is expressed, as we have already observed, by a fraction.

Thus, if we have to divide 1 by $1 - a$, we obtain the fraction $\frac{1}{1-a}$. This, however, does not prevent us from attempting the division, according to the rules that have been given, and continuing it as far as we please. We shall not fail to find the true quotient, though under different forms.

249. To prove this, let us actually divide the dividend 1 by the divisor $1-a$, thus:

$$1-a)\ 1\ (1+\frac{a}{1-a};\ \text{or,}\ 1-a)\ 1\ (1+a+\frac{aa}{1-a}$$

$$\begin{array}{ll} \underline{1-a} & \quad\underline{1-a} \\ \text{remainder } a & \quad a \\ & \quad \underline{a-aa} \\ & \text{remainder } aa \end{array}$$

To find a greater number of forms, we have only to continue dividing aa by $1-a$;

$$1-a)\ aa\ (aa+\frac{a^3}{1-a},\ \text{then}\ 1-a)\ a^3\ (a^3+\frac{a^4}{1-a}$$

$$\begin{array}{ll} \underline{aa-a^3} & \quad \underline{a^3-a^4} \\ a^3 & \quad a^4 \end{array}$$

and again $1-a)\ a^4\ (a^4+\dfrac{a^5}{1-a}$

$$\begin{array}{l} \underline{a^4-a^5} \\ a^5,\ \&c. \end{array}$$

250. This shews that the fraction $\dfrac{1}{1-a}$ may be exhibited under all the following forms:

$$\text{I.)}\ 1+\frac{a}{1-a};\quad \text{II.)}\ 1+a+\frac{aa}{1-a};$$

$$\text{III.)}\ 1+a+aa+\frac{a^3}{1-a};\quad \text{IV.)}\ 1+a+aa+a^3+\frac{a^4}{1-a};$$

$$\text{V.)}\ 1+a+aa+a^3+a^4+\frac{a^5}{1-a},\ \&c.$$

Now, by considering the first of these expressions, which is $1+\dfrac{a}{1-a}$, and remembering that 1 is the same as $\dfrac{1-a}{1-a}$, we have

$$1+\frac{a}{1-a}=\frac{1-a}{1-a}+\frac{a}{1-a}=\frac{1-a+a}{1-a}=\frac{1}{1-a}.$$

If we follow the same process with regard to the second expression $1+a+\dfrac{aa}{1-a}$, that is to say, if we reduce the integral part $1+a$ to the same denominator $1-a$, we shall have $\frac{1-aa}{1-a}$, to which if we add $+\frac{aa}{1-a}$, we shall have $\frac{1-aa+aa}{1-a}$, that is to say, $\frac{1}{1-a}$.

In the third expression, $1+a+aa+\frac{a^3}{1-a}$, the integers reduced to the denominator $1-a$ make $\frac{1-a^3}{1-a}$; and if we add to that the fraction $\frac{a^3}{1-a}$, we have $\frac{1}{1-a}$; wherefore all these expressions are equal in value to $\frac{1}{1-a}$, the proposed fraction.

251. This being the case, we may continue the series as far as we please, without being under the necessity of performing any more calculations. We shall therefore have

$$\frac{1}{1-a}=1+a+aa+a^3+a^4+a^5+a^6+a^7+\frac{a^8}{1-a};$$

or we might continue this further, and still go on without end. For this reason, it may be said, that the proposed fraction has been resolved into an infinite series, which is

$1+a+aa+a^3+a^4+a^5+a^6+a^7+a^8+a^9+a^{10}+a^{11}+a^{12}$, &c. to infinity. And there are sufficient grounds to maintain, that the value of this infinite series is the same as that of the fraction $\frac{1}{1-a}$.

252. What we have said may, at first, appear surprising; but the consideration of some particular cases will make it easily understood.

Let us suppose, in the first place, $a=1$; our series will become $1+1+1+1+1+1+1$, &c. The fraction $\frac{1}{1-a}$, to which it must be equal, becomes $\frac{1}{0}$. Now, we before remarked, that $\frac{1}{0}$ is a number infinitely great; which is, therefore, here confirmed in a satisfactory manner.

But if we suppose $a = 2$, our series becomes $= 1 + 2 + 4 + 8 + 16 + 32 + 64$, &c. to infinity, and its value must be $\frac{1}{1-2}$, that is to say, $\frac{1}{-1} = -1$; which at first sight will appear absurd. But it must be remarked, that if we wish to stop at any term of the above series, we cannot do so without joining the fraction which remains. Suppose, for example, we were to stop at 64, after having written $1 + 2 + 4 + 8 + 16 + 32 + 64$, we must join the fraction $\frac{128}{1-2}$, or $\frac{128}{-1}$, or -128; we shall therefore have $127 - 128$, that is in fact -1.

Were we to continue the series without intermission, the fraction indeed would be no longer considered, but then the series would still go on.

253. These are the considerations which are necessary, when we assume for a numbers greater than unity. But if we suppose a less than 1, the whole becomes more intelligible.

For example, let $a = \frac{1}{2}$; we shall have

$$\frac{1}{1-a} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2,$$

which will be equal to the following series:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \tfrac{1}{128}, \text{ \&c. to infinity.}$$

Now, if we take only two terms of this series, we have $1 + \frac{1}{2}$, and it wants $\frac{1}{2}$, that it may be equal to $\frac{1}{1-a} = 2$. If we take three terms, it wants $\frac{1}{4}$; for the sum is $1\frac{3}{4}$. If we take four terms we have $1\frac{7}{8}$, and the deficiency is only $\frac{1}{8}$. We see, therefore, that the more terms we take, the less the difference becomes, and that, consequently, if we continue on to infinity, there will be no difference at all between the sum of the series and 2, the value of the fraction $\frac{1}{1-a}$.

254. Let $a = \frac{1}{3}$; our fraction $\frac{1}{1-a}$ will be $= \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1\frac{1}{2}$, which, reduced to an infinite series, becomes

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \tfrac{1}{243}, \text{ \&c.}$$

and to which $\frac{1}{1-a}$ is consequently equal.

When we take two terms, we have $1\frac{1}{3}$, and there wants $\frac{1}{6}$. If we take three terms, we have $1\frac{4}{9}$, and there will still be wanting $\frac{1}{18}$. Take four terms, we shall have $1\frac{13}{27}$, and the difference is $\frac{1}{54}$. Since the error, therefore, always becomes three times less, it must evidently vanish at last.

255. Suppose $a = \frac{2}{3}$; we shall have $\frac{1}{1-a} = \frac{1}{1-\frac{2}{3}} = 3$, and the series $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243}$, &c. to infinity. Taking first $1\frac{2}{3}$, the error is $1\frac{1}{3}$; taking three terms, which make $2\frac{1}{9}$, the error is $\frac{8}{9}$; taking four terms we have $2\frac{11}{27}$, and the error is $\frac{16}{27}$.

256. If $a = \frac{1}{4}$, the fraction is $\frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = 1\frac{1}{3}$; and the series becomes $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$, &c. The two first terms making $1 + \frac{1}{4}$, will give $\frac{1}{12}$ for the error; and taking one term more, we have $1\frac{5}{15}$, that is to say, only an error of $\frac{1}{48}$.

257. In the same manner, we may resolve the fraction $\frac{1}{1+a}$, into an infinite series by actually dividing the numerator 1 by the denominator $1+a$, as follows:

$$
\begin{array}{rlllll}
1+a) & 1 & (1 - a + aa - a^3 + a^4 & & & \\
 & \underline{1+a} & & & & \\
 & -a & & & & \\
 & \underline{-a - aa} & & & & \\
 & & aa & & & \\
 & & \underline{aa + a^3} & & & \\
 & & & -a^3 & & \\
 & & & \underline{-a^3 - a^4} & & \\
 & & & & a^4 & \\
 & & & & \underline{a^4 + a^5} & \\
 & & & & -a^5, \text{ \&c.} &
\end{array}
$$

Whence it follows, that the fraction $\frac{1}{1+a}$ is equal to the series,

$$1 - a + aa - a^3 + a^4 - a^5 + a^6 - a^7, \text{ \&c.}$$

258. If we make $a = 1$, we have this remarkable comparison: $\frac{1}{1+a} = \frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1$, &c. to infinity. This will appear rather contradictory; for if we stop at -1, the series gives 0; and if we finish by $+1$, it gives 1. But this is precisely what solves the difficulty; for since we must go on to infinity without stopping either at -1, or at $+1$, it is evident that the sum can neither be 0 nor 1, but that this result must lie between these two, and therefore be $= \frac{1}{2}$.

259. Let us now make $a = \frac{1}{2}$, and our fraction will be $\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$, which must therefore express the value of the series

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \tfrac{1}{32} + \tfrac{1}{64}, \text{ \&c. to infinity.}$$

If we take only the two leading terms of this series, we have $\frac{1}{2}$, which is too small by $\frac{1}{6}$. If we take three terms, we have $\frac{3}{4}$, which is too much by $\frac{1}{12}$. If we take four terms, we have $\frac{5}{8}$ which is too small by $\frac{1}{24}$, &c.

260. Suppose again $a = \frac{1}{3}$; our fraction will be $= \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$, and to this the series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729}$, &c. continued to infinity, must be equal. Now, by considering only two terms, we have $\frac{2}{3}$, which is too small by $\frac{1}{12}$. Three terms make $\frac{7}{9}$, which is too much by $\frac{1}{36}$. Four terms make $\frac{20}{27}$, which is too small by $\frac{1}{108}$, and so on.

261. The fraction $\frac{1}{1+a}$ may also be resolved into an infinite series another way; namely, by dividing 1 by $a + 1$, as follows:

$$
\begin{array}{l}
a+1)\ 1\ (\frac{1}{a}-\frac{1}{aa}+\frac{1}{a^3}-\frac{1}{a^4}+\frac{1}{a^5} \\
\quad 1+\frac{1}{a} \\
\hline
\quad\quad -\frac{1}{a} \\
\quad\quad -\frac{1}{a}-\frac{1}{aa} \\
\hline
\quad\quad\quad \frac{1}{aa} \\
\quad\quad\quad \frac{1}{aa}+\frac{1}{a^3} \\
\hline
\quad\quad\quad\quad -\frac{1}{a^3} \\
\quad\quad\quad\quad -\frac{1}{a^3}-\frac{1}{a^4} \\
\hline
\quad\quad\quad\quad\quad \frac{1}{a^4} \\
\quad\quad\quad\quad\quad \frac{1}{a^4}+\frac{1}{a^5} \\
\hline
\quad\quad\quad\quad\quad\quad -\frac{1}{a^5}, \text{ \&c.}
\end{array}
$$

Consequently, our fraction $\frac{1}{a+1}$, is equal to the infinite series $\frac{1}{a}-\frac{1}{aa}+\frac{1}{a^3}-\frac{1}{a^4}+\frac{1}{a^5}-\frac{1}{a^6}$, &c. Let us make $a=1$, and we shall have the series

$$1-1+1-1+1-1, \text{ \&c.} = \tfrac{1}{2}, \text{ as before.}$$

And if we suppose $a=2$, we shall have the series

$$\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{8}-\tfrac{1}{16}+\tfrac{1}{32}-\tfrac{1}{64}, \text{ \&c.} = \tfrac{1}{3}.$$

262. In the same manner, by resolving the general fraction $\frac{c}{a+b}$ into an infinite series, we shall have,

$$
\begin{array}{l}
a+b)\; c\; \left(\frac{c}{a}-\frac{b\,c}{a\,a}+\frac{b\,b\,c}{a^3}-\frac{b^3\,c}{a^4}\right.\\
\qquad c+\frac{b\,c}{a}\\
\qquad \overline{\qquad\qquad}\\
\qquad -\frac{b\,c}{a}\\
\qquad -\frac{b\,c}{a}-\frac{b\,b\,c}{a\,a}\\
\qquad \overline{\qquad\qquad\qquad}\\
\qquad\qquad\quad \frac{b\,b\,c}{a\,a}\\
\qquad\qquad\quad \frac{b\,b\,c}{a\,a}+\frac{b^3\,c}{a^3}\\
\qquad\qquad\quad \overline{\qquad\qquad\qquad}\\
\qquad\qquad\qquad -\frac{b^3\,c}{a^3}\\
\qquad\qquad\qquad -\frac{b^3\,c}{a^3}-\frac{b^4\,c}{a^4}\\
\qquad\qquad\qquad \overline{\qquad\qquad\qquad}\\
\qquad\qquad\qquad\qquad\quad \frac{b^4\,c}{a^4};
\end{array}
$$

Whence it appears, that we may compare $\frac{c}{a+b}$ with the series $\frac{c}{a}-\frac{b\,c}{a\,a}+\frac{b\,b\,c}{a^3}-\frac{b^3\,c}{a^4}$, &c. to infinity.

Let $a=2$, $b=4$, $c=3$, and we shall have

$$\frac{c}{a+b}=\frac{3}{2+4}=\tfrac{3}{6}=\tfrac{1}{2}=\tfrac{3}{2}-3+6-12, \text{ \&c.}$$

Let $a=10$, $b=1$, and $c=11$, and we have

$$\frac{c}{a+b}=\frac{11}{10+1}=1=\tfrac{11}{10}-\tfrac{11}{100}+\tfrac{11}{1000}-\tfrac{11}{10000}, \text{ \&c.}$$

If we consider only one term of this series, we have $\frac{11}{10}$, which is too much by $\frac{1}{10}$; if we take two terms, we have $\frac{99}{100}$, which is too small by $\frac{1}{100}$; if we take three terms, we have $\frac{1001}{1000}$, which is too much by $\frac{1}{1000}$, &c.

263. When there are more than two terms in the divisor, we may also continue the division to infinity in the same manner.

Thus, if the fraction $\frac{1}{1-a+a\,a}$ were proposed, the infinite series, to which it is equal, would be found as follows:

$$
\begin{array}{llllll}
1-a+aa)\ 1 & (1+a-a^3-a^4+a^6+a^7,\ \&c. \\
\quad 1-a+aa & \\
\hline
\quad\quad a-aa & \\
\quad\quad a-aa+a^3 & \\
\hline
\quad\quad\quad -a^3 & \\
\quad\quad\quad -a^3+a^4-a^5 & \\
\hline
\quad\quad\quad\quad -a^4+a^5 & \\
\quad\quad\quad\quad -a^4+a^5-a^6 & \\
\hline
\quad\quad\quad\quad\quad a^6 & \\
\quad\quad\quad\quad\quad a^6-a^7+a^8 & \\
\hline
\quad\quad\quad\quad\quad\quad a^7-a^8 & \\
\quad\quad\quad\quad\quad\quad a^7-a^8+a^9 & \\
\hline
\quad\quad\quad\quad\quad\quad\quad -a^9 &
\end{array}
$$

We have therefore the equation of

$$\frac{1}{1-a+aa}=1+a-a^3-a^4+a^6+a^7-a^9-a^{10},\ \&c.$$

Here, if we make $a=1$, we have

$$1=1+1-1-1+1+1-1-1+1+1,\ \&c.$$

which series contains twice the series found above

$$1-1+1-1+1,\ \&c.$$

Now, as we have found this $=\frac{1}{2}$, it is not astonishing that we should find $\frac{2}{2}$, or 1, for the value of that which we have just determined.

Make $a=\frac{1}{2}$, and we shall then have the equation

$$\frac{1}{\frac{3}{4}}=\frac{4}{3}=1+\frac{1}{2}-\frac{1}{8}-\frac{1}{16}+\frac{1}{64}+\frac{1}{128}-\frac{1}{512},\ \&c.$$

Suppose $a=\frac{1}{3}$, we shall have the equation

$$\frac{1}{\frac{7}{9}}=\frac{9}{7}=1+\frac{1}{3}-\frac{1}{27}-\frac{1}{81}+\frac{1}{729},\ \&c.$$

If we take the four leading terms of this series, we have $\frac{104}{81}$, which is only $\frac{1}{567}$, less than $\frac{9}{7}$.

Suppose again $a=\frac{2}{3}$, we shall have

$$\frac{1}{\frac{7}{9}}=\frac{9}{7}=1+\frac{2}{3}-\frac{8}{27}-\frac{16}{81}+\frac{64}{729},\ \&c.$$

This series must therefore be equal to the preceding one; and subtracting one from the other, $\frac{1}{3}-\frac{7}{27}-\frac{15}{81}+\frac{63}{29}$, must be $=0$. These four terms added together make $-\frac{2}{81}$.

264. The method, which we have explained, serves to resolve, generally, all fractions into infinite series; and, therefore, it is often found to be of the greatest utility. Further, it is remarkable, that *an infinite series, though it never ceases, may have a determinate value.* It may be added, that from this branch of mathematics inventions of the utmost importance have been derived, on which account the subject deserves to be studied with the greatest attention.

CHAPTER VI.

Of the Squares of Compound Quantities.

265. WHEN it is required to find the square of a compound quantity, we have only to multiply it by itself, and the product will be the square required.

For example, the square of $a+b$ is found in the following manner:

$$\begin{array}{l} a+b \\ a+b \\ \hline aa+ab \\ \qquad ab+bb \\ \hline aa+2ab+bb. \end{array}$$

266. So that, *when the root consists of two terms added together,* as $a+b$, *the square comprehends,* 1*st, the square of each term,* namely, $a\,a$ and $b\,b$; 2*dly, twice the product of the two terms,* namely, $2\,a\,b$. So that the sum $a\,a+2\,a\,b+b\,b$ is the square of $a+b$. Let, for example, $a=10$ and $b=3$, that is to say, let it be required to find the square of 13, we shall have $100+60+9$, or 169.

267. We may easily find, by means of this formula, the squares of numbers, however great, if we divide them into two parts. To find, for example, the square of 57, we consider that this number is $=50+7$; whence we conclude that its square is $=2500+700+49=3249$.

268. Hence it is evident, that the square of $a+1$ will be $a\,a+2\,a+1$: now since the square of a is $a\,a$, we find the square

$a+1$ by adding to that $2a+1$; and it must be observed, that this $2a+1$ is the sum of the two roots a and $a+1$.

Thus, as the square of 10 is 100, that of 11 will be $100+21$. The square of 57 being 3249, that of 58 is $3249+115=3364$. The square of $59=3364+117=3481$; the square of

$$60=3481+119=3600, \text{ \&c.}$$

269. The square of a compound quantity, as $a+b$, is represented in this manner: $(a+b)^2$. We have then

$$(a+b)^2=aa+2ab+bb,$$

whence we deduce the following equations:

$$(a+1)^2=aa+2a+1;\ (a+2)^2=aa+4a+4;$$
$$(a+3)^2=aa+6a+9;\ (a+4)^2=aa+8a+16;\ \text{\&c.}$$

270. *If the root is* a — b, *the square of it is* a a — 2 a b + b b, *which contains also the squares of the two terms, but in such a manner that we must take from their sum twice the product of those two terms.*

Let, for example, $a=10$ and $b=-1$, the square of 9 will be found $=100-20+1=81$.

271. Since we have the equation $(a-b)^2=aa-2ab+bb$, we shall have $(a-1)^2=aa-2a+1$. *The square of* a — 1 *is found, therefore, by subtracting from* a a *the sum of the two roots* a *and* a — 1, *namely,* 2 a — 1. Let, for example, $a=50$, we have $aa=2500$, and $a-1=49$: then $49^2=2500-99=2401$.

272. What we have said may be also confirmed and illustrated by fractions. For if we take as the root $\frac{3}{5}+\frac{2}{5}$ (which make 1) the squares will be:

$$\tfrac{9}{25}+\tfrac{4}{25}+\tfrac{12}{25}=\tfrac{25}{25}, \text{ that is } 1,$$

Further, the square of $\frac{1}{2}-\frac{1}{3}$ (or of $\frac{1}{6}$) will be

$$\tfrac{1}{4}-\tfrac{1}{3}+\tfrac{1}{9}=\tfrac{1}{36}.$$

273. When the root consists of a greater number of terms, the method of determining the square is the same. Let us find, for example, *the square of* a + b + c.

$$\begin{array}{l}
a+b+c \\
a+b+c \\
\hline
aa+ab+ac \qquad +bc \\
\qquad ab+ac+bb+bc+cc \\
\hline
aa+2ab+2ac+bb+2bc+cc.
\end{array}$$

We see that it *includes, first, the square of each term of the root, and beside that, the double products of those terms multiplied two by two.*

274. To illustrate this by an example, let us divide the number 256 into three parts, $200 + 50 + 6$; its square will then be composed of the following parts :

$$
\begin{array}{r}
40000 \\
2500 \\
36 \\
20000 \\
2400 \\
600 \\
\hline
65536
\end{array}
\qquad\qquad\qquad
\begin{array}{r}
256 \\
256 \\
\hline
1536 \\
1280 \\
512 \\
\hline
65536
\end{array}
$$

which is evidently equal to the product of 256×256.

275. *When some terms of the root are negative, the square is still found by the same rule ; but we must take care what signs we prefix to the double products.* Thus, the square of $a - b - c$ being $aa + bb + cc - 2ab - 2ac + 2bc$. if we represent the number 256 by $300 - 40 - 4$, we shall have,

$$
\underbrace{\begin{array}{r}
+90000 \\
1600 \\
320 \\
16 \\
\hline
+91936 \\
-26400 \\
\hline
\end{array}}_{}
\qquad\qquad
\begin{array}{r}
-24000 \\
-\ \ 2400 \\
\hline
-26400
\end{array}
$$

Positive Parts. | Negative Parts.

65536, the square of 256, as before.

CHAPTER VII.

Of the Extraction of Roots applied to Compound Quantities.

276. In order to give a certain rule for this operation, we must consider attentively the square of the root $a+b$, which is $aa+2ab+bb$, that we may reciprocally find the root of a given square.

277. We must consider therefore, first, that as the square $aa+2ab+bb$ is composed of several terms, it is certain that the root also will comprise more than one term; and that if we write the square, in such a manner that the powers of one of the letters, as a, may go on continually diminishing, the first term will be the square of the first term of the root. And since, in the present case, the first term of the square is aa, it is certain that the first term of the root is a.

278. Having, therefore, found the first term of the root, that is to say a, we must consider the rest of the square, namely, $2ab+bb$, to see if we can derive from it the second part of the root, which is b. Now this remainder $2ab+bb$ may be represented by the product, $(2a+b)b$. Wherefore the remainder having two factors, $2a+b$ and b, it is evident that we shall find the latter, b, which is the second part of the root, by dividing the remainder $2ab+bb$ by $2a+b$.

279. So that the quotient, arising from the division of the above remainder by $2a+b$, is the second term of the root required. Now, in this division we observe, that $2a$ is the double of the first term a, which is already determined. So that although the second term is yet unknown, and it is necessary, for the present, to leave its place empty, we may nevertheless attempt the division, since in it we attend only to the first term $2a$. But as soon as the quotient is found, which is here b, we must put it in the empty place, and thus render the division complete.

280. The calculation, therefore, by which we find the root of the square $aa+2ab+bb$, may be represented thus:

$$\begin{array}{rl} & aa+2ab+bb\,(a+b \\ & aa \\ \cline{2-2} 2a+b) & 2ab+bb \\ & 2ab+bb \\ \cline{2-2} & \quad 0. \end{array}$$

281. We may, in the same manner, find the square root of other compound quantities, provided they are squares, as the following examples will shew.

$$\begin{array}{rl} & aa+6ab+9bb\,(a+3b \\ & aa \\ \cline{2-2} 2a+3b) & 6ab+9bb \\ & 6ab+9bb \\ \cline{2-2} & \quad 0. \end{array}$$

$$\begin{array}{rl} & 4aa-4ab+bb\,(2a-b \\ & 4aa \\ \cline{2-2} 4a-b) & -4ab+bb \\ & -4ab+bb \\ \cline{2-2} & \quad 0. \end{array}$$

$$\begin{array}{rl} & 9pp+24pq+16qq\,(3p+4q \\ & 9pp \\ \cline{2-2} 6p+4q) & 24pq+16qq \\ & 24pq+16qq \\ \cline{2-2} & \quad 0. \end{array}$$

$$\begin{array}{rl} & 25xx-60x+36\,(5x-6 \\ & 25xx \\ \cline{2-2} 10x-6) & -60x+36 \\ & -60x+36 \\ \cline{2-2} & \quad 0. \end{array}$$

282. When there is a remainder after the division, it is a proof that the root is composed of more than two terms. We then consider the two terms already found as forming the first part, and endeavour to derive the other from the remainder, in the same manner as we found the second term of the root. The following examples will render this operation more clear.

$$
\begin{array}{rl}
 & aa+2ab-2ac-2bc+bb+cc\,(a+b-c \\
 & aa \\ \hline
2a+b) & 2ab-2ac-2bc+bb+cc \\
 & 2ab \qquad\qquad\qquad +bb \\ \hline
2a+2b-c) & -2ac-2bc+cc \\
 & -2ac-2bc+cc \\ \hline
 & 0.
\end{array}
$$

$$
\begin{array}{rl}
 & a^4+2a^3+3aa+2a+1\,(aa+a+1 \\
 & a^4 \\ \hline
2aa+a) & 2a^3+3aa \\
 & 2a^3+\ aa \\ \hline
2aa+2a+1) & 2aa+2a+1 \\
 & 2aa+2a+1 \\ \hline
 & 0.
\end{array}
$$

$$
\begin{array}{rl}
 & a^4-4a^3b+8ab^3+4b^4\,(aa-2ab-2bb \\
 & a^4 \\ \hline
2aa-2ab) & -4a^3b+8ab^3+4b^4 \\
 & -4a^3b+4aabb \\ \hline
2aa-4ab-2bb) & -4aabb+8ab^3+4b^4 \\
 & -4aabb+8ab^3+4b^4 \\ \hline
 & 0.
\end{array}
$$

$$\begin{array}{rl}
 & a^6 - 6a^5b + 15a^4bb - 20a^3b^3 + 15aab^4 - 6ab^5 + b^6 \quad (a^3 - 3aab + 3abb - b^3 \\
 & a^6 \\ \hline
2a^3 - 3aab) & -6a^5b + 15a^4bb \\
 & -6a^5b + 9a^4bb \\ \hline
2a^3 - 6aab + 3abb) & 6a^4bb - 20a^3b^3 + 15aab^4 \\
 & 6a^4bb - 18a^3b^3 + 9aab^4 \\ \hline
2a^3 - 6aab + 6abb - b^3) & -2a^3b^3 + 6aab^4 - 6ab^5 + b^6 \\
 & -2a^3b^3 + 6aab^4 - 6ab^5 + b^6 \\ \hline
 & 0.
\end{array}$$

283. We easily deduce from the rule which we have explained, the method which is taught in books of arithmetic for the extraction of the square root. Some examples in numbers :

$$\begin{array}{rl}
 & \dot{5}2\dot{9}\ (23 \\
 & 4 \\ \hline
43) & 129 \\
 & 129 \\ \hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & 1\dot{7}6\dot{4}\ (42 \\
 & 16 \\ \hline
82) & 164 \\
 & 164 \\ \hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & 2\dot{3}0\dot{4}\ (48 \\
 & 16 \\ \hline
88) & 704 \\
 & 704 \\ \hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & 4\dot{0}9\dot{6}\ (64 \\
 & 36 \\ \hline
124) & 496 \\
 & 496 \\ \hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & 9\dot{6}0\dot{4}\ (98 \\
 & 81 \\ \hline
188) & 1504 \\
 & 1504 \\ \hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & \dot{1}5\dot{6}2\dot{5}\ (125 \\
 & 1 \\ \hline
22) & 56 \\
 & 44 \\ \hline
245) & 1225 \\
 & 1225 \\ \hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & 9\dot{9}8\dot{0}0\dot{1}\ (999 \\
 & 81 \\ \hline
189) & 1880 \\
 & 1701 \\ \hline
1989) & 17901 \\
 & 17901 \\ \hline
 & 0.
\end{array}$$

284. But when there is a remainder after the whole operation, it is a proof that the number proposed is not a square, and consequently that its root cannot be assigned. In such cases, the radical sign, which we before employed, is made use of. It is written before the quantity, and the quantity itself is placed between parentheses, or under a line. Thus, the square root of $aa+bb$ is represented by $\sqrt{(aa+bb)}$, or by $\sqrt{\overline{aa+bb}}$; and $\sqrt{(1-xx)}$, or $\sqrt{\overline{1-xx}}$, expresses the square root of $1-xx$. Instead of this radical sign, we may use the fractional exponent $\frac{1}{2}$, and represent the square root of $aa+bb$, for instance, by $(aa+bb)^{\frac{1}{2}}$, or by $\overline{aa+bb}\rceil^{\frac{1}{2}}$.

CHAPTER VIII.

Of the calculation of Irrational Quantities.

285. When it is required to add together two or more irrational quantities, this is done, according to the method before laid down, by writing all the terms in succession, each with its proper sign. And with regard to abbreviation, we must remark that *instead of* $\sqrt{a}+\sqrt{a}$, for example, *we write* $2\sqrt{a}$; *and* that $\sqrt{a}-\sqrt{a}=0$, because these two terms destroy one another. Thus, *the quantities* $3+\sqrt{2}$ *and* $1+\sqrt{2}$, *added together, make* $4+2\sqrt{2}$ *or* $4+\sqrt{8}$; the sum of $5+\sqrt{3}$ and $4-\sqrt{3}$ is 9; and that of $2\sqrt{3}+3\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ is $3\sqrt{3}+2\sqrt{2}$.

286. Subtraction also is very easy, since we have only to add the proposed numbers, changing first their signs: the following example will shew this: let us subtract the lower number from the upper.

$$\begin{array}{l} 4-\sqrt{2}+2\sqrt{3}-3\sqrt{5}+4\sqrt{6} \\ 1+2\sqrt{2}-2\sqrt{3}-5\sqrt{5}+6\sqrt{6} \\ \hline 3-3\sqrt{2}+4\sqrt{3}+2\sqrt{5}-2\sqrt{6} \end{array}$$

287. In multiplication we must recollect that $\sqrt{a}$ *multiplied by* $\sqrt{a}$ *produces* a; and that *if the numbers which follow the sign* $\surd$ *are different, as* a *and* b, *we have* $\sqrt{ab}$ *for the product of* $\sqrt{a}$ *multiplied by* $\sqrt{b}$. After this it will be easy to perform the following examples:

$$\begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad +\sqrt{2}+2 \\ \hline 1+2\sqrt{2}+2=3+2\sqrt{2} \end{array} \qquad \begin{array}{l} 4+2\sqrt{2} \\ 2-\ \ \sqrt{2} \\ \hline 8+4\sqrt{2} \\ \quad -4\sqrt{2}-4 \\ \hline 8-4=4 \end{array}$$

288. What we have said applies also to imaginary quantities; we shall only observe further, that $\sqrt{-a}$ *multiplied by* $\sqrt{-a}$ *produces* $-a$.

If it were required to find the cube of $-1+\sqrt{-3}$, we should take the square of that number, and then multiply that square by the same number: see the operation:

$$\begin{array}{l} -1+\sqrt{-3} \\ -1+\sqrt{-3} \\ \hline \quad 1-\sqrt{-3} \\ \quad\quad -\sqrt{-3}-3 \\ \hline 1-2\sqrt{-3}-3=-2-2\sqrt{-3} \\ \qquad\qquad\qquad\quad -1+\ \ \sqrt{-3} \\ \hline \qquad\qquad\qquad\quad\ 2+2\sqrt{-3} \\ \qquad\qquad\qquad\qquad -2\sqrt{-3}+6 \\ \hline \qquad\qquad\qquad\quad 2+6=8. \end{array}$$

289. *In the division of surds, we have only to express the proposed quantities in the form of a fraction; this may be then changed into another expression having a rational denominator.* For if the denominator be $a+\sqrt{b}$, for example, and we multiply both it and the numerator by $a-\sqrt{b}$, the new denominator will be $aa-b$, in which there is no radical sign. Let it be proposed to divide $3+2\sqrt{2}$ by $1+\sqrt{2}$; we shall first have $\frac{3+2\sqrt{2}}{1+\ \ \sqrt{2}}$. Multiplying now the two terms of the fraction by $1-\sqrt{2}$, we shall have for the numerator:

$$\begin{array}{l} 3+2\sqrt{2} \\ 1-\ \sqrt{2} \\ \hline 3+2\sqrt{2} \\ \quad -3\sqrt{2}-4 \\ \hline 3-\ \sqrt{2}-4=-\sqrt{2}-1; \end{array}$$

and for the denominator:

$$\begin{array}{l} 1+\sqrt{2} \\ 1-\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad -\sqrt{2}-2 \\ \hline 1-2=-1 \end{array}$$

Our new fraction therefore is $\dfrac{-\sqrt{2}-1}{-1}$; and if we again multiply the two terms by -1, we shall have for the numerator $\sqrt{2}+1$, and for the denominator $+1$. Now it is easy to shew that $\sqrt{2}+1$ is equal to the proposed fraction $\dfrac{3+2\sqrt{2}}{1+\ \sqrt{2}}$; for $\sqrt{2}+1$ being multiplied by the divisor $1+\sqrt{2}$, thus,

$$\begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad +\sqrt{2}+2 \\ \hline \end{array}$$

we have $1+2\sqrt{2}+2=3+2\sqrt{2}$.

Another example: $8-5\sqrt{2}$ divided by $3-2\sqrt{2}$ makes $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$. Multiplying the two terms of this fraction by $3+2\sqrt{2}$, we have for the numerator,

$$\begin{array}{l} 8-5\sqrt{2} \\ 3+2\sqrt{2} \\ \hline 24-15\sqrt{2} \\ \quad +16\sqrt{2}-20 \\ \hline 24+\sqrt{2}-20=4+\sqrt{2}; \end{array}$$

and for the denominator,

$$\begin{array}{l} 3 - 2\sqrt{2} \\ 3 + 2\sqrt{2} \\ \hline 9 - 6\sqrt{2} \\ \quad + 6\sqrt{2} - 8 \\ \hline 9 - 8 = +1. \end{array}$$

Consequently the quotient will be $4 + \sqrt{2}$. The truth of this may be proved in the following manner:

$$\begin{array}{l} 4 + \sqrt{2} \\ 3 - 2\sqrt{2} \\ \hline 12 + 3\sqrt{2} \\ \quad - 8\sqrt{2} - 4 \\ \hline 12 - 5\sqrt{2} - 4 = 8 - 5\sqrt{2} \end{array}$$

290. In the same manner, we may transform such fractions into others, that have rational denominators. If we have, for example, the fraction $\frac{1}{5 - 2\sqrt{6}}$, and multiply its numerator and denominator by $5 + 2\sqrt{6}$, we transform it into this

$$\frac{5 + 2\sqrt{6}}{1} = 5 + 2\sqrt{6}.$$

In like manner, the fraction $\frac{2}{-1 + \sqrt{-3}}$ assumes this form,

$$\frac{2 + 2\sqrt{-3}}{-4} = \frac{1 + \sqrt{-3}}{-2}.$$

And $\frac{\sqrt{6} + \sqrt{5}}{\sqrt{6} - \sqrt{5}}$ becomes $= \frac{11 + 2\sqrt{30}}{1} = 11 + 2\sqrt{30}$.

291. *When the denominator contains several terms, we may in the same manner make the radical signs in it vanish one by one.* Let the fraction $\frac{1}{\sqrt{10} - \sqrt{2} - \sqrt{3}}$ be proposed; we first multiply these terms by $\sqrt{10} + \sqrt{2} + \sqrt{3}$, and obtain the fraction $\frac{\sqrt{10} + \sqrt{2} + \sqrt{3}}{5 - 2\sqrt{6}}$. Then multiplying its numerator and denominator by $5 + 2\sqrt{6}$, we have $5\sqrt{10} + 11\sqrt{2} + 9\sqrt{3} + 2\sqrt{60}$.

CHAPTER IX.

Of Cubes, and the Extraction of Cube Roots.

292. To find *the cube of a root* a + b, we only multiply its square $a a + 2 a b + b b$ again by $a + b$, thus,

$$\begin{array}{l} a a + 2 a b + b b \\ \quad a + b \\ \hline a^3 + 2 a a b + a b b \\ \qquad\quad a a b + 2 a b b + b^3 \\ \hline \end{array}$$

and the cube will be $= a^3 + 3 a a b + 3 a b b + b^3$.

It *contains*, therefore, *the cubes of the two parts of the root, and* beside that, $3 a a b + 3 a b b$, a quantity equal to $(3 a b) \times (a + b)$; that is, *the triple product of the two parts*, a *and* b, *multiplied by their sum.*

293. So that whenever a root is composed of two terms, it is easy to find its cube by this rule. For example, the number $5 = 3 + 2$; its cube is therefore $27 + 8 + 18 \times 5 = 125$.

Let $7 + 3 = 10$ be the root; the cube will be

$$343 + 27 + 63 \times 10 = 1000.$$

To find the cube of 36, let us suppose the root $36 = 30 + 6$, and we have for the power required,

$$27000 + 216 + 540 \times 36 = 46656.$$

294. But if, on the other hand, the cube be given, namely, $a^3 + 3 a a b + 3 a b b + b^3$, and it be required to find its root, we must premise the following remarks:

First, when the cube is arranged according to the powers of one letter, we easily know by the first term a^3, the first term a of the root, since the cube of it is a^3; if, therefore, we subtract that cube from the cube proposed, we obtain the remainder, $3 a a b + 3 a b b + b^3$, which must furnish the second term of the root.

295. But as we already know that the second term is $+ b$, we have principally to discover how it may be derived from the above remainder. Now that remainder may be expressed by two factors, as $(3 a a + 3 a b + b b) \times (b)$; if, therefore, we divide

by $3aa + 3ab + bb$, we obtain the second part of the root $+ b$, which is required.

296. But as this second term is supposed to be unknown, the divisor also is unknown; nevertheless we have the first term of that divisor, which is sufficient; for it is $3aa$, that is, thrice the square of the first term already found; and by means of this, it is not difficult to find also the other part, b, and then to complete the divisor before we perform the division. For this purpose, it will be necessary to join to $3aa$ thrice the product of the two terms, or $3ab$, and bb, or the square of the second term of the root.

297. Let us apply what we have said to two examples of other given cubes.

I.

$$\begin{array}{rl}
 & a^3 + 12aa + 48a + 64\ (a + 4 \\
 & a^3 \\
\hline
3aa + 12a + 16) & 12aa + 48a + 64 \\
 & 12aa + 48a + 64 \\
\hline
 & 0.
\end{array}$$

II.

$$\begin{array}{rl}
 & a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1 \\
 & a^6 \qquad\qquad\qquad\qquad (aa - 2a + 1 \\
\hline
3a^4 - 6a^3 + 4aa) & -6a^5 + 15a^4 - 20a^3 \\
 & -6a^5 + 12a^4 - 8a^3 \\
\hline
3a^4 - 12a^3 + 12aa + 3a^2 - 6a + 1) & 3a^4 - 12a^3 + 15aa - 6a + 1 \\
 & 3a^4 - 12a^3 + 15aa - 6a + 1 \\
\hline
 & 0.
\end{array}$$

298. The analysis which we have given is the foundation of the common rule for the extraction of the cube root in numbers. An example of the operation in the number 2197:

$$\begin{array}{r|l}
 & \dot{2}19\dot{7}\ (10 + 3 = 13 \\
 & 1000 \\
\hline
300 & 1197 \\
90 & \\
9 & \\
\hline
399 & 1197 \\
\hline
 & 0.
\end{array}$$

Let us also extract the cube root of 34965783 :

	34965783	(300 + 20 + 7
	27000000	
270000	7965783	
18000		
400		
288400	5768000	
307200	2197783	
6720		
49		
313969	2197783	
	0.	

CHAPTER X.

Of the higher Powers of Compound Quantities.

299. After squares and cubes come higher powers, or powers of greater number of degrees. They are represented by exponents in the manner which we before explained : we have only to remember, when the root is compound, to inclose it in a parenthesis. Thus $(a+b)^5$ means that $a+b$ is raised to the fifth degree, and $(a-b)^6$ represents the sixth power of $a-b$. We shall in this chapter explain the nature of these powers.

300. Let $a+b$ be the root, or the first power, and the higher powers will be found by multiplication in the following manner :

$$
\begin{array}{rl}
(a+b)^1 = & a + b \\
 & a + b \\
\hline
 & a^2 + ab \\
 & \quad + ab + bb \\
\hline
(a+b)^2 = & a^2 + 2ab + bb \\
 & a + b \\
\hline
 & a^3 + 2aab + abb \\
 & \quad + aab + 2abb + b^3 \\
\hline
(a+b)^3 = & a^3 + 3aab + 3abb + b^3 \\
 & a + b \\
\hline
 & a^4 + 3a^3b + 3aabb + ab^3 \\
 & \quad + a^3b + 3aabb + 3ab^3 + b^4 \\
\hline
(a+b)^4 = & a^4 + 4a^3b + 6aabb + 4ab^3 + b^4 \\
 & a + b \\
\hline
 & a^5 + 4a^4b + 6a^3bb + 4aab^3 + ab^4 \\
 & \quad + a^4b + 4a^3bb + 6aab^3 + 4ab^4 + b^5 \\
\hline
(a+b)^5 = & a^5 + 5a^4b + 10a^3bb + 10aab^3 + 5ab^4 + b^5 \\
 & a + b \\
\hline
 & a^6 + 5a^5b + 10a^4bb + 10a^3b^3 + 5aab^4 + ab^5 \\
 & \quad + a^5b + 5a^4bb + 10a^3b^3 + 10aab^4 + 5ab^5 + b^6 \\
\hline
(a+b)^6 = & a^6 + 6a^5b + 15a^4bb + 20a^3b^3 + 15aab^4 + 6ab^5 + b^6
\end{array}
$$

301. The powers of the root $a - b$ are found in the same manner, and we shall immediately perceive that they do not differ from the preceding, excepting that the 2d, 4th, 6th, &c. terms are affected by the sign *minus* ;

$$
\begin{array}{l}
(a-b)^1 = a-b \\
\qquad\qquad a-b \\
\hline
\qquad\qquad aa-b \\
\qquad\qquad\quad -ab+bb \\
\hline
(a-b)^2 = a^2-2ab+bb \\
\qquad\qquad a\ -b \\
\hline
\qquad\qquad a^3-2aab+abb \\
\qquad\qquad\quad -\ aab+2abb-b^3 \\
\hline
(a-b)^3 = a^3-3auo+3abb-b^3 \\
\qquad\qquad a\ -b \\
\hline
\qquad\qquad a^4-3a^3b+3aabb-ab^3 \\
\qquad\qquad\quad -\ a^3b+3aabb-3ab^3+b^4 \\
\hline
(a-b)^4 = a^4-4a^3b+6aabb-4ab^3+b^4 \\
\qquad\qquad a\ -b \\
\hline
\qquad\qquad a^5-4a^4b+6a^3bb-4aab^3+ab^4 \\
\qquad\qquad\quad -\ a^4b+4a^3bb-6aab^3+4ab^4-b^5 \\
\hline
(a-b)^5 = a^5-5a^4b+10a^3bb-10aab^3+5ab^4-b^5 \\
\qquad\qquad a\ -b \\
\hline
\qquad\qquad a^6-5a^5b+10a^4bb-10a^3b^3+5aab^4-ab^5 \\
\qquad\qquad\quad -\ a^5b+5a^4bb-10a^3b^3+10aab^4-5ab^5+b^6 \\
\hline
(a-b)^6 = a^6-6a^5b+15a^4bb-20a^3b^3+15aab^4-6ab^5+b^6.
\end{array}
$$

Here we see that all the odd powers of b have the sign —, while the even powers retain the sign +. The reason of this is evident; for since $-b$ is a term of the root, the powers of that letter will ascend in the following series, $-b$, $+bb$, $-b^3$, $+b^4$, $-b^5$, $+a^6$, &c. which clearly shews that the even powers must be affected by the sign +, and the odd ones by the contrary sign —.

302. An important question occurs in this place; namely, how we may find, without being obliged always to perform the same calculation, all the powers either of $a+b$, or $a-b$.

We must remark, in the first place, that if we can assign all the powers of $a+b$, those of $a-b$ are also found, since we have only to change the signs of the even terms, that is to say, of the second, the fourth, the sixth, &c. The business then is to establish a rule, by which *any power of* a + b, *however high, may be determined without the necessity of calculating all the preceding ones.*

303. Now, if from the powers which we have already determined we take away the numbers that precede each term, which are called the *coefficients*, we observe in all the terms a singular order; *first, we see the first term* a *of the root raised to the power which is required; in the following terms the powers of* a *diminish continually by unity, and the powers of* b *increase in the same proportion; so that the sum of the exponents of* a *and of* b *is always the same, and always equal to the exponent of the power required; and, lastly, we find the term* b *by itself raised to the same power.* If, therefore, the tenth power of $a+b$ were required, we are certain that the terms, without their coefficients would succeed each other in the following order; a^{10}, a^9b, a^8b^2, a^7b^3, a^6b^4, a^5b^5, a^4b^6, a^3b^7, a^2b^8, ab^9, b^{10}.

304. It remains, therefore, to shew how we are to determine the coefficients which belong to those terms, or the numbers by which they are to be multiplied. Now, *with respect to the first term, its coefficient is always unity; and with regard to the second, its coefficient is constantly the exponent of the power;* but with regard to the other terms, it is not so easy to observe any order in their coefficients. However, if we continue those coefficients, we shall not fail to discover a law, by which we may advance as far as we please. This the following table will shew.

Powers.	Coefficients.
I.	1, 1
II.	1, 2, 1
III.	1, 3, 3, 1
IV.	1, 4, 6, 4, 1
V.	1, 5, 10, 10, 5, 1
VI.	1, 6, 15, 20, 15, 6, 1
VII.	1, 7, 21, 35, 35, 21, 7, 1
VIII.	1, 8, 28, 56, 70, 56, 28, 8, 1
IX.	1, 9, 36, 84, 126, 126, 84, 36, 9, 1
X.	1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1, &c.

We see then, that the tenth power of $a + b$ will be $a^{10} + 10a^9b + 45a^8bb + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45aab^8 + 10ab^9 + b^{10}$.

305. *With regard to the coefficients, it must be observed, that for each power their sum must be equal to the number 2 raised to the same power.* Let $a = 1$ and $b = 1$, each term, without the coefficients, will be $= 1$; consequently, the value of the power will be simply the sum of the coefficients; this sum, in the preceding example, is 1024, and accordingly

$$(1 + 1)^{10} = 2^{10} = 1024.$$

It is the same with respect to other powers; we have for the

I. $1 + 1 = 2 = 2^1$,

II. $1 + 2 + 1 = 4 = 2^2$,

III. $1 + 3 + 3 + 1 = 8 = 2^3$,

IV. $1 + 4 + 6 + 4 + 1 = 16 = 2^4$,

V. $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$

VI. $1 + 6 + 15 + 20 + 15 + 6 + 1 = 64 = 2^6$

VII. $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128 = 2^7$, &c.

306. Another necessary remark, with regard to the coefficients, is, that they increase from the beginning to the middle, and then decrease in the same order. In the even powers, the greatest coefficient is exactly in the middle; but in the odd powers, two coefficients, equal and greater than the others, are found in the middle, belonging to the mean terms.

The order of the coefficients deserves particular attention; for it is in this order that we discover the means of determining them for any power whatever, without calculating all the preceding powers. We shall explain this method, reserving the demonstration however for the next chapter.

307. *In order to find the coefficients of any power proposed, the seventh, for example, let us write the following fractions, one after the other;*

$$\tfrac{7}{1},\ \tfrac{6}{2},\ \tfrac{5}{3},\ \tfrac{4}{4},\ \tfrac{3}{5},\ \tfrac{2}{6},\ \tfrac{1}{7}.$$

In this arrangement we perceive that the numerators begin by the exponent of the power required, and that they diminish successively by unity; while the denominators follow in the natural order of the numbers, 1, 2, 3, 4, &c. Now, the first coefficient being always 1, the first fraction gives the second coefficient. The product of the two first fractions, multiplied together, represents the third coefficient. The product of the three first fractions represents the fourth coefficient, and so on.

So that the first coefficient $=1$; the second $=\frac{7}{1}=7$; the third $=\frac{7}{1}\times\frac{6}{2}=21$; the fourth $=\frac{7}{1}\times\frac{6}{2}\times\frac{5}{3}=35$; the fifth $=\frac{7}{1}\times\frac{6}{2}\times\frac{5}{3}\times\frac{4}{4}=35$; the sixth $=\frac{7}{1}\times\frac{6}{2}\times\frac{5}{3}\times\frac{4}{4}\times\frac{3}{5}=21$; the seventh $=21\times\frac{2}{6}=7$; the eighth $=7\times\frac{1}{7}=1$.

308. So that we have, for the second power, the two fractions $\frac{2}{1}, \frac{1}{2}$; whence it follows, that the first coefficient $=1$; the second $=\frac{2}{1}=2$; and the third $=2\times\frac{1}{2}=1$.

The third power furnishes the fractions $\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$; wherefore the first coefficient $=1$; the second $=\frac{3}{1}=3$; the third $=3\times\frac{2}{2}=3$; the fourth $=\frac{3}{1}\times\frac{2}{2}\times\frac{1}{3}=1$.

We have for the fourth power, the fractions $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}$; consequently the first coefficient $=1$; the second $\frac{4}{1}=4$; the third $\frac{4}{1}\times\frac{3}{2}=6$; the fourth $\frac{4}{1}\times\frac{3}{2}\times\frac{2}{3}=4$; and the fifth $\frac{4}{1}\times\frac{3}{2}\times\frac{2}{3}\times\frac{1}{4}=1$.

309. This rule evidently renders it unnecessary for us to find the preceding coefficients, and enables us to discover immediately the coefficients which belong to any power. Thus, for the tenth power, we write the fractions $\frac{10}{1}, \frac{9}{2}, \frac{8}{3}, \frac{7}{4}, \frac{6}{5}, \frac{5}{6}, \frac{4}{7}, \frac{3}{8}, \frac{2}{9}, \frac{1}{10}$, by means of which we find

the first coefficient $=1$,
the second $=\frac{10}{1}=10$,
the third $=10\times\frac{9}{2}=45$,
the fourth $=45\times\frac{8}{3}=120$,
the fifth $=120\times\frac{7}{4}=210$,

the sixth	$=210 \times \frac{6}{5} = 252,$
the seventh	$=252 \times \frac{5}{6} = 210,$
the eighth	$=210 \times \frac{4}{7} = 120,$
the ninth	$=120 \times \frac{3}{8} = 45,$
the tenth	$= 45 \times \frac{2}{9} = 10,$
the eleventh	$= 10 \times \frac{1}{10} = 1.$

310. We may also write these fractions as they are, without computing their value; and in this way it is easy to express any power of $a+b$, however high. Thus, the hundredth power of $a+b$ will be $(a+b)^{100} = a^{100} + \frac{100}{1} \times a^{99}b + \frac{100 \times 99}{1 \times 2} + a^{98}b^2 + \frac{100 \times 99 \times 98}{1 \times 2 \times 3} a^{97}b^3 + \frac{100 \times 99 \times 98 \times 97}{1 \times 2 \times 3 \times 4} a^{96}b^4 +,$ &c.. whence the law of the succeeding terms may be easily deduced.

CHAPTER XI.

Of the Transposition of the Letters, on which the demonstration of the preceding rule is founded.

311. If we trace back the origin of the coefficients which we have been considering, we shall find, that each term is presented, as many times as it is possible to transpose the letters, of which that term consists; or, to express the same thing differently, the coefficient of each term is equal to the number of transpositions that the letters admit, of which that term is composed. In the second power, for example, the term $a\,b$ is taken twice, that is to say, its coefficient is 2; and in fact we may change the order of the letters which compose that term twice, since we may write $a\,b$ and $b\,a$; the term $a\,a$, on the contrary, is found only once, because the order of the letters can undergo no change, or transposition. In the third power of $a+b$, the term $a\,a\,b$ may be written in three different ways. $a\,a\,b$, $a\,b\,a$, $b\,a\,a$; thus the coefficient is 3. Likewise, in the fourth power, the term $a^3\,b$ or $a\,a\,a\,b$, admits of four different arrangements. $a\,a\,a\,b$, $a\,a\,b\,a$, $a\,b\,a\,a$, $b\,a\,a\,a$; therefore its coefficient is 4. The term $a\,a\,b\,b$ admits of six transpositions, $a\,a\,b\,b$, $a\,b\,b\,a$, $b\,a\,b\,a$, $a\,b\,a\,b$, $b\,b\,a\,a$, $b\,a\,a\,b$, and its coefficient is 6. It is the same in all cases.

312. In fact, if we consider that the fourth power, for example, of any root consisting of more than two terms, as $(a+b+c+d)^4$, is found by multiplying the four factors, I. $a+b+c+d$; II. $a+b+c+d$; III. $a+b+c+d$; IV. $a+b+c+d$; we me may easily see, that each letter of the first factor must be multiplied by each letter of the second, then by each letter of the third, and, lastly, by each letter of the fourth.

Each term must therefore not only be composed of four letters, but also present itself, or enter into the sum, as many times as those letters can be differently arranged with respect to each other, whence arises its coefficient.

313. It is therefore of great importance to know, in how many different ways a given number of letters may be arranged. And, in this inquiry, we must particularly consider, whether the letters in question are the same, or different. When they are the same, there can be no transposition of them, and for this reason the simple powers, as a^2, a^3, a^4, &c., have all unity for the coefficient.

314. Let us first suppose all the letters different; and beginning with the simplest case of two letters, or ab, we immediately discover that two transpositions may take place, namely, ab and ba.

If we have three letters abc, to consider, we observe that each of the three may take the first place, while the two others will admit of two transpositions. For if a is the first letter, we have two arrangements, abc, acb; if b is in the first place, we have the arrangements bac, bca; lastly, if c occupies the first place, we have also two arrangements, namely, cab, cba. And consequently the whole number of arrangements is $3 \times 2 = 6$.

If there are four letters, $abcd$, each may occupy the first place; and in each case the three others may form six different arrangements, as we have just seen. The whole number of transpositions is therefore $4 \times 6 = 24 = 4 \times 3 \times 2 \times 1$.

If there are five letters, $abcde$, each of the five must be the first, and the four others will admit of twenty-four transpositions; so that the whole number of transpositions will be $5 \times 24 = 120 = 5 \times 4 \times 3 \times 2 \times 1$.

315. Consequently, however great the number of letters may be, it is evident, provided they are all different, that we may

easily determine the number of transpositions, and that we may make use of the following table :

Number of Letters.	Number of Transpositions.
I.	$1 = 1.$
II.	$2 \times 1 = 2.$
III.	$3 \times 2 \times 1 = 6.$
IV.	$4 \times 3 \times 2 \times 1 = 24.$
V.	$5 \times 4 \times 3 \times 2 \times 1 = 120.$
VI.	$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$
VII.	$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$
VIII.	$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320.$
IX.	$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880.$
X.	$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800.$

316. But, as we have intimated, the numbers in this table can be made use of only when all the letters are different; for if two or more of them are alike, the number of transpositions becomes much less ; and if all the letters are the same, we have only one arrangement. We shall now see how the numbers in the table are to be diminished, according to the number of letters that are alike.

317. When two letters are given, and those letters are the same, the two arrangements are reduced to one, and consequently the number, which we have found above, is reduced to the half ; that is to say, it must be divided by 2. If we have three letters alike, the six transpositions are reduced to one ; whence it follows that the numbers in the table must be divided by $6 = 3 \times 2 \times 1$. And for the same reason, if four letters are alike, we must divide the numbers found by 24 or $4 \times 3 \times 2 \times 1$, &c.

It is easy therefore to determine how many transpositions the letters *a a a b b c*, for example, may undergo. They are in number 6, and consequently, if they were all different, they would admit of $6 \times 5 \times 4 \times 3 \times 2 \times 1$ transpositions. But since *a* is found thrice in those letters, we must divide that number of transpositions by $3 \times 2 \times 1$; and since *b* occurs twice, we must again divide it by 2×1; the number of transpositions required will therefore be $= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 5 \times 4 \times 3 = 60.$

318. It will now be easy for us to determine the coefficients of all the terms of any power. We shall give an example of the seventh power $(a+b)^7$.

The first term is a^7, which occurs only once; and as all the other terms have each seven letters, it follows that the number of transpositions for each term would be $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, if all the letters were different. But since in the second term, $a^6 b$, we find six letters alike, we must divide the above product by $6 \times 5 \times 4 \times 3 \times 2 \times 1$, whence it follows that the coefficient is $= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{7}{1}$.

In the third term $a^5\,b\,b$, we find the same letter a five times, and the same letter b twice; we must therefore divide that number first by $5 \times 4 \times 3 \times 2 \times 1$, and then also by 2×1; whence results the coefficient $\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1} = \frac{7 \times 6}{2 \times 1}$.

The fourth term $a^4\,b^3$ contains the letter a four times, and the letter b thrice; consequently, the whole number of the transpositions of the seven letters must be divided, in the first place, by $4 \times 3 \times 2 \times 1$, and secondly, by $3 \times 2 \times 1$, and the coefficient becomes $= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} = \frac{7 \times 6 \times 5}{1 \times 2 \times 3}$.

In the same manner, we find $\frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4}$ for the coefficient of the fifth term, and so of the rest; by which the rule before given is demonstrated.

319. These considerations carry us further, and shew us also, how to find all the powers of roots composed of more than two terms. We shall apply them to the third power of $a+b+c$; the terms of which must be formed by all the possible combinations of three letters, each term having for its coefficient the number of its transpositions, as above.

Without performing the multiplication, the third power of $(a+b+c)$ will be $a^3 + 3\,a\,a\,b + 3\,a\,a\,c + 3\,a\,b\,b + 6\,a\,b\,c + 3\,a\,c\,c + b^3 + 3\,b\,b\,c + 3\,b\,c\,c + c^3$.

Suppose $a=1$, $b=1$, $c=1$, the cube of $1+1+1$, or of 3, will be $1+3+3+3+6+3+1+3+3+1=27$.

This result is accurate, and confirms the rule.

If we had supposed $a = 1$, $b = 1$, and $c = -1$, we should have found for the cube of $1 + 1 - 1$, that is, of 1,

$$1 + 3 - 3 + 3 - 6 + 3 + 1 - 3 + 3 - 1 = 1.$$

CHAPTER XII.

Of the expression of Irrational Powers by Infinite Series.

320. As we have shewn the method of finding any power of the root $a + b$, however great the exponent, we are able to express, generally, the power of $a + b$, whose exponent is undetermined. It is evident that if we represent that exponent by n, we shall have by the rule already given (art. 307 and the following):

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1} b + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} b^4 \text{ \&c.}$$

321. If the same power of the root $a - b$ were required, we should only change the signs of the second, fourth, sixth, &c. terms, and should have $(a - b)^n = a^n - \frac{n}{1} a^{n-1} b + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} b^4$, &c.

322. These formulas are remarkably useful; for they serve also to express all kinds of radicals. We have shewn that all irrational quantities may assume the form of powers, whose exponents are fractional, and that $\sqrt[2]{a} = a^{\frac{1}{2}}$; $\sqrt[3]{a} = a^{\frac{1}{3}}$, and $\sqrt[4]{a} = a^{\frac{1}{4}}$, &c. We have therefore, also,

$$\sqrt[2]{(a + b)} = (a + b)^{\frac{1}{2}};\ \sqrt[3]{(a + b)} = (a + b)^{\frac{1}{3}}$$

$$\text{and } \sqrt[4]{(a + b)} = (a + b)^{\frac{1}{4}}, \text{ \&c.}$$

Wherefore, if we wish to find the square root of $a + b$, we have only to substitute for the exponent n the fraction $\frac{1}{2}$, in the general formula, [art. 320,] and we shall have first, for the coefficients,

$\frac{n}{1} = \frac{1}{2}$; $\frac{n-1}{2} = -\frac{1}{4}$; $\frac{n-2}{3} = -\frac{3}{6}$; $\frac{n-3}{4} = -\frac{5}{8}$; $\frac{n-4}{5} = -\frac{7}{10}$; $\frac{n-5}{6} = -\frac{9}{12}$. Then, $a^n = a^{\frac{1}{2}} = \sqrt{a}$ and $a^{n-1} = \frac{1}{\sqrt{a}}$; $a^{n-2} = \frac{1}{a\sqrt{a}}$; $a^{n-3} = \frac{1}{aa\sqrt{a}}$, &c., or we might express those powers of a in the following manner; $a^n = \sqrt{a}$; $a^{n-1} = \frac{a^n}{a} = \frac{\sqrt{a}}{a}$; $a^{n-2} = \frac{a^n}{a^2} = \frac{\sqrt{a}}{a^2}$; $a^{n-3} = \frac{a^n}{a^3} = \frac{\sqrt{a}}{a^3}$; $a^{n-4} = \frac{a^n}{a^4} = \frac{\sqrt{a}}{a^4}$, &c.

323. This being laid down, the square root of $a + b$, may be expressed in the following manner:

$$\sqrt{(a+b)} = \sqrt{a} + \frac{1}{2}b\frac{\sqrt{a}}{a} - \frac{1}{2}\times\frac{1}{4}bb\frac{\sqrt{a}}{aa} + \frac{1}{2}\times\frac{1}{4}\times\frac{3}{6}b^3\frac{\sqrt{a}}{a^3} - \frac{1}{2}\times\frac{1}{4}\times\frac{3}{6}\times\frac{5}{8}b^4\frac{\sqrt{a}}{a^4}, \text{ \&c.}$$

324. If a, therefore, be a square number, we may assign the value of $\sqrt{a}$, and, consequently, the square root of $a + b$ may be expressed by an infinite series, without any radical sign.

Let, for example, $a = cc$, we shall have $\sqrt{a} = c$; then

$$\sqrt{(cc+b)} = c + \frac{1}{2}\times\frac{b}{c} - \frac{1}{8}\frac{bb}{c^3} + \frac{1}{16}\times\frac{b^3}{c^5} - \frac{5}{128}\times\frac{b^4}{c^7}, \text{ \&c.}$$

We see, therefore, that there is no number, whose square root we may not extract in the same way; since every number may be resolved into two parts, one of which is a square represented by cc. If we require, for example, the square root of 6, we make $6 = 4 + 2$, consequently $cc = 4$, $c = 2$, $b = 2$, whence results $\sqrt{6} = 2 + \frac{1}{2} - \frac{1}{16} + \frac{1}{64} - \frac{5}{1024}$, &c.

If we take only the two leading terms of this series, we shall have $2\frac{1}{2} = \frac{5}{2}$, the square of which, $\frac{25}{4}$, is $\frac{1}{4}$ greater than 6; but if we consider three terms, we have $2\frac{7}{16} = \frac{39}{16}$, the square of which, $\frac{1521}{256}$, is still $\frac{15}{256}$ too small.

325. Since, in this example, $\frac{5}{2}$ approaches very nearly to the true value of $\sqrt{6}$, we shall take for 6 the equivalent quantity $\frac{25}{4} - \frac{1}{4}$. Thus $cc = \frac{25}{4}$; $c = \frac{5}{2}$; $b = -\frac{1}{4}$; and calculating only the two leading terms, we find $\sqrt{6} = \frac{5}{2} + \frac{1}{2}\times\frac{-\frac{1}{4}}{\frac{5}{2}} = \frac{5}{2} - \frac{1}{2}\times\frac{\frac{1}{4}}{\frac{5}{2}}$

$= \frac{5}{2} - \frac{1}{20} = \frac{49}{20}$: the square of this fraction, being $\frac{2401}{400}$, exceeds the square of $\sqrt{6}$ only by $\frac{1}{400}$.

Now, making $6 = \frac{2401}{400} - \frac{1}{400}$, so that $c = \frac{49}{20}$ and $b = -\frac{1}{400}$; and still taking only the two leading terms, we have

$$\sqrt{6} = \frac{49}{20} + \frac{1}{2} \times \frac{-\frac{1}{400}}{\frac{49}{20}} = \frac{49}{20} - \frac{1}{2} \times \frac{\frac{1}{400}}{\frac{49}{20}} = \frac{49}{20} - \frac{1}{1960} = \frac{4801}{1960},$$

the square of which is $\frac{23049601}{3841600}$. Now 6, when reduced to the same denominator, is $= \frac{23049600}{3841600}$; the error therefore is only $\frac{1}{3841600}$.

326. In the same manner, we may express the cube root of $a + b$ by an infinite series. For since $\sqrt[3]{(a+b)} = (a+b)^{\frac{1}{3}}$, we shall have in the general formula $n = \frac{1}{3}$, and for the coefficients, $\frac{n}{1} = \frac{1}{3}$; $\frac{n-1}{2} = -\frac{1}{3}$; $\frac{n-2}{3} = -\frac{5}{9}$; $\frac{n-3}{4} = -\frac{2}{3}$; $\frac{n-4}{5} = -\frac{11}{15}$, &c., and with regard to the powers of a, we shall have $a^n = \sqrt[3]{a}$; $a^{n-1} = \frac{\sqrt[3]{a}}{a}$; $a^{n-2} = \frac{\sqrt[3]{a}}{aa}$; $a^{n-3} = \frac{\sqrt[3]{a}}{a^3}$, &c.; then

$$\sqrt[3]{(a+b)} = \sqrt[3]{a} + \frac{1}{3} \times b\frac{\sqrt[3]{a}}{a} - \frac{1}{9} \times bb\frac{\sqrt[3]{a}}{aa} + \frac{5}{81} \times b^3\frac{\sqrt[3]{a}}{a^3} - \frac{10}{243} \times b^4\frac{\sqrt[3]{a}}{a^4}, \&c.$$

327. If a therefore be a cube, or $a = c^3$, we have $\sqrt[3]{a} = c$, and the radical signs will vanish; for we shall have

$$\sqrt[3]{(c^3+b)} = c + \frac{1}{3} \times \frac{b}{cc} - \frac{1}{9} \times \frac{bb}{c^5} + \frac{5}{81} \times \frac{b^3}{c^8} - \frac{10}{243} \times \frac{b^4}{c^{11}} \&c.$$

328. We have, therefore, arrived at a formula, which will enable us to find *by approximation*, as it is called, the cube root of any number; since every number may be resolved into two parts, as $c^3 + b$, the first of which is a cube.

If we wish, for example, to determine the cube root of 2, we represent 2 by $1 + 1$, so that $c = 1$ and $b = 1$, consequently $\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81}$, &c., the two leading terms of this series make $1\frac{1}{3} = \frac{4}{3}$ the cube of which, $\frac{64}{27}$, is too great by $\frac{10}{27}$. Let us then make $2 = \frac{64}{27} - \frac{10}{27}$, we have $c = \frac{4}{3}$ and $b = -\frac{10}{27}$, and consequently $\sqrt[3]{2} = \frac{4}{3} + \frac{1}{3} \times \frac{-\frac{10}{27}}{\frac{16}{9}}$. These two terms give $\frac{4}{3} - \frac{5}{72} = \frac{91}{72}$, the cube of which is $\frac{753571}{373248}$. Now, $2 = \frac{746496}{373248}$, so that the error is $\frac{7075}{373248}$. In this way we might still approximate, and the faster in proportion as we take a greater number of terms.

CHAPTER XIII.

Of the resolution of Negative Powers.

329. We have already shewn, that we may express $\frac{1}{a}$ by a^{-1}; we may therefore also express $\frac{1}{a+b}$ by $(a+b)^{-1}$; so that the fraction $\frac{1}{a+b}$ may be considered as a power of $a+b$, namely, that power whose exponent is -1; and from this it follows, that the series already found as the value of $(a+b)^n$ extends also to this case.

330. Since, therefore, $\frac{1}{a+b}$ is the same as $(a+b)^{-1}$, let us suppose, in the general formula, $n = -1$; and we shall first have for the coefficients $\frac{n}{1} = -1$; $\frac{n-1}{2} = -1$; $\frac{n-2}{3} = -1$; $\frac{n-3}{4} = -1$, &c. Then, for the powers of a; $a^n = a^{-1} = \frac{1}{a}$; $a^{n-1} = a^{-2} = \frac{1}{a^2}$; $a^{n-2} = \frac{1}{a^3}$; $a^{n-3} = \frac{1}{a^4}$, &c. So that $(a+b)^{-1}$ $= \frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + \frac{bb}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6}$, &c., and this is the same series that we found before by division.

331. Further, $\frac{1}{(a+b)^2}$ being the same with $(a+b)^{-2}$, let us reduce this quantity also to an infinite series. For this purpose, we must suppose $n = -2$, and we shall first have for the coeffi-

cients $\frac{n}{1} = -\frac{2}{1}$; $\frac{n-1}{2} = -\frac{3}{2}$; $\frac{n-2}{3} = -\frac{4}{3}$; $\frac{n-3}{4} = -\frac{5}{4}$, &c. Then, for the powers of a; $a^n = \frac{1}{a^2}$; $a^{n-1} = \frac{1}{a^3}$; $a^{n-2} = \frac{1}{a^4}$; $a^{n-3} = \frac{1}{a^5}$, &c. We therefore obtain $(a+b)^{-2} = \frac{1}{(a+b)^2} = \frac{1}{a^2} - \frac{2}{1} \times \frac{b}{a^3} + \frac{2}{1} \times \frac{3}{2} \times \frac{bb}{a^4} - \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \frac{b^3}{a^5} + \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \frac{b^4}{a^6}$, &c. Now, $\frac{2}{1} = 2$; $\frac{2}{1} \times \frac{3}{2} = 3$; $\frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} = 4$; $\frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} = 5$, &c. Consequently, we have $\frac{1}{(a+b)^2} = \frac{1}{a^2} - 2\frac{b}{a^3} + 3\frac{b^2}{a^4} - 4\frac{b^3}{a^5} + 5\frac{b^4}{a^6} - 6\frac{b^5}{a^7} + 7\frac{b^6}{a^8}$, &c.

332. Let us proceed and suppose $n = -3$, and we shall have a series expressing the value of $\frac{1}{(a+b)^3}$, or of $(a+b)^{-3}$. The coefficients will be $\frac{n}{1} = -\frac{3}{1}$; $\frac{n-1}{2} = -\frac{4}{2}$; $\frac{n-2}{3} = -\frac{5}{3}$; $\frac{n-3}{4} = -\frac{6}{4}$, &c. and the powers of a become, $a^n = \frac{1}{a^3}$; $a^{n-1} = \frac{1}{a^4}$; $a^{n-2} = \frac{1}{a^5}$, &c., which gives $\frac{1}{(a+b)^3} = \frac{1}{a^3} - \frac{3}{1}\frac{b}{a^4} + \frac{3}{1} \times \frac{4}{2}\frac{b^2}{a^5} - \frac{3}{1} \times \frac{4}{2} \times \frac{5}{3}\frac{b^3}{a^6} + \frac{3}{1} \times \frac{4}{2} \times \frac{5}{3} \times \frac{6}{4}\frac{b^4}{a^7}$, &c. $= \frac{1}{a^3} - 3\frac{b}{a^4} + 6\frac{b^2}{a^5} - 10\frac{b^3}{a^6} + 15\frac{b^4}{a^7} - 21\frac{b^5}{a^8} + 28\frac{b^6}{a^9} - 36\frac{b^7}{a^{10}} + 45\frac{b^8}{a^{11}}$, &c.

Let us now make $n = -4$; we shall have for the coefficients $\frac{n}{1} = -\frac{4}{1}$; $\frac{n-1}{2} = -\frac{5}{2}$; $\frac{n-2}{3} = -\frac{6}{3}$; $\frac{n-3}{4} = -\frac{7}{4}$, &c., and for the powers, $a^n = \frac{1}{a^4}$; $a^{n-1} = \frac{1}{a^5}$; $a^{n-2} = \frac{1}{a^6}$; $a^{n-3} = \frac{1}{a^7}$; $a^{n-4} = \frac{1}{a^8}$, &c., whence we obtain; $\frac{1}{(a+b)^4} = \frac{1}{a^4} - \frac{4}{1} \times \frac{b}{a^5} + \frac{4}{1} \times \frac{5}{2} \times \frac{b^2}{a^6} - \frac{4}{1} \times \frac{5}{2} \times \frac{6}{3} \times \frac{b^3}{a^7} + \frac{4}{1} \times \frac{5}{2} \times \frac{6}{3} \times \frac{7}{4} \times \frac{b^4}{a^8}$; &c.

$$= \frac{1}{a^4} - 4\frac{b}{a^5} + 10\frac{b^2}{a^6} - 20\frac{b^3}{a^7} + 35\frac{b^4}{a^8} - 56\frac{b^5}{a^9} \text{ \&c.}$$

333. The different cases that have been considered enable us to conclude, with certainty, that we shall have, generally, for any negative power of $a+b$;

$$\frac{1}{(a+b)^m} = \frac{1}{a^m} - \frac{m}{1} \times \frac{b}{a^{m+1}} + \frac{m}{1} \times \frac{m+1}{2} \times \frac{b^2}{a^{m+2}} - \frac{m}{1} \times \frac{m+1}{2} \times$$
$$\frac{m+2}{3} \times \frac{b^3}{a^{m+3}} \text{ \&c.}$$

And by means of this formula, we may transform all such fractions into infinite series, substituting fractions also, or fractional exponents, for m, in order to express irrational quantities.

334. The following considerations will illustrate this subject further.

We have seen that,

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6} \text{ \&c.}$$

If, therefore, we multiply this series by $a+b$, the product ought to be $=1$; and this is found to be true, as we shall see by performing the multiplication:

$$\begin{array}{l}
\frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6} +, \text{\&c.} \\
a+b \\
\hline
1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{b^3}{a^3} + \frac{b^4}{a^4} - \frac{b^5}{a^5} +, \text{\&c.} \\
\quad + \frac{b}{a} - \frac{b^2}{a^2} + \frac{b^3}{a^3} - \frac{b^4}{a^4} + \frac{b^5}{a^5} -, \text{\&c.} \\
\hline
1.
\end{array}$$

335. We have also found, that

$$\frac{1}{(a+b)^2} = \frac{1}{aa} - \frac{2b}{a^3} + \frac{3bb}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6} - \frac{6b^5}{a^7}, \text{ \&c.}$$

If, therefore, we multiply this series by $(a+b)^2$, the product ought also to be $=1$. Now $(a+b)^2 = aa + 2ab + bb$. See the operation:

$$\frac{1}{aa} - \frac{2b}{a^3} + \frac{3bb}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6} - \frac{6b^5}{a^7} +, \&c.$$
$$aa + 2ab + bb$$

$$1 - \frac{2b}{a} + \frac{3bb}{aa} - \frac{4b^3}{a^3} + \frac{5b^4}{a^4} - \frac{6b^5}{a^5} +, \&c.$$
$$+ \frac{2b}{a} - \frac{4bb}{aa} + \frac{6b^3}{a^3} - \frac{8b^4}{a^4} + \frac{10b^5}{a^5} -, \&c.$$
$$+ \frac{bb}{aa} - \frac{2b^3}{a^3} + \frac{3b^4}{a^4} - \frac{4b^5}{a^5} +, \&c.$$

$1 =$ the product, which the nature of the thing required.

336. If we multiply the series which we found for the value of $\frac{1}{(a+b)^2}$, by $a+b$ only, the product ought to answer to the fraction $\frac{1}{a+b}$, or be equal to the series already found, namely, $\frac{1}{a} - \frac{b}{a^2} + \frac{bb}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5}$, &c. and this the actual multiplication will confirm.

$$\frac{1}{aa} - \frac{2b}{a^3} + \frac{3bb}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6}, \&c.$$
$$a + b$$

$$\frac{1}{a} - \frac{2b}{aa} + \frac{3bb}{a^3} - \frac{4b^3}{a^4} + \frac{5b^4}{a^5}, \&c.$$
$$+ \frac{b}{aa} - \frac{2bb}{a^3} + \frac{3b^3}{a^4} - \frac{4b^4}{a^5}, \&c.$$

$$\frac{1}{a} - \frac{b}{aa} + \frac{bb}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} -, \&c.$$

SECTION III.

OF RATIOS AND PROPORTIONS.

CHAPTER. I.

Of Arithmetical Ratio, or of the difference between two Numbers.

ARTICLE 337.

Two quantities are either equal to one another, or they are not. In the latter case, where one is greater than the other, we may consider their inequality in two different points of view: we may ask, *how much* one of the quantities is greater than the other? Or, we may ask, *how many times* the one is greater than the other? The results, which constitute the answers to these two questions, are both called *relations* or *ratios*. We usually call the former *arithmetical ratio*, and the latter *geometrical ratio*, without however these denominations having any connexion with the thing itself: they have been adopted arbitrarily.

338. It is evident, that the quantities of which we speak must be of one and the same kind; otherwise, we could not determine any thing with regard to their equality or inequality. It would be absurd, for example, to ask if two pounds and three ells are equal quantities. So that in what follows, quantities of the same kind only are to be considered; and as they may always be expressed by numbers, it is of numbers only, as was mentioned at the beginning, that we shall treat.

339. When of two given numbers, therefore, it is required to find, how much one is greater than the other, the answer to this question determines the arithmetical ratio of the two numbers. Now, since this answer consists in giving the difference of the

two numbers, it follows, that an arithmetical ratio is nothing but the *difference* between two numbers: and as this appears to be a better expression, we shall reserve the words *ratio* and *relation*, to express geometrical ratios.

340. The difference between two numbers is found, we know, by subtracting the less from the greater; nothing therefore can be easier than resolving the question, how much one is greater than the other. So that when the numbers are equal, the difference being nothing, if it be inquired how much one of the numbers is greater than the other, we answer, by nothing. For example, 6 being $= 2 \times 3$, the difference between 6 and 2×3 is 0.

341. But when the two numbers are not equal, as 5 and 3, and it is inquired how much 5 is greater than 3, the answer is, 2; and it is obtained by subtracting 3 from 5. Likewise 15 is greater than 5 by 10; and 20 exceeds 8 by 12.

342. We have three things, therefore, to consider on this subject; 1st, the greater of the two numbers; 2d, the less; and 3d, the difference. And these three quantities are connected together in such a manner, that two of the three being given, we may always determine the third.

Let the greater number $= a$, the less $= b$, and the difference $= d$; the difference d will be found by subtracting b from a, so that $d = a - b$; whence we see how to find d, when a and b are given.

343. But if the difference and the less of the two numbers, or b, are given, we can determine the greater number by adding together the difference and the less number, which gives $a = b + d$. For, if we take from $b + d$ the less number b, there remains d, which is the known difference. Let the less number $= 12$, and the difference $= 8$; then the greater number will be $= 20$.

344. Lastly, if beside the difference d, the greater number a is given, the other number b is found by subtracting the difference from the greater number, which gives $b = a - d$. For if I take the number $a - d$ from the greater number a, there remains d, which is the given difference.

345. The connexion, therefore, among the numbers a, b, d, is of such a nature, as to give the three following results: 1st. $d = a$

$-b$; 2[d.] $a=b+d$; 3[d.] $b=a-d$; and if one of these three comparisons be just, the others must necessarily be so also. Wherefore, generally, if $z=x+y$, it necessarily follows, that $y=z-x$. and $x=z-y$.

346. With regard to these arithmetical ratios we must remark, that *if we add to the two numbers* a *and* b, *a number* c *assumed at pleasure, or subtract it from them, the difference remains the same.* That is to say, if d is the difference between a and b, that number d will also be the difference between $a+c$ and $b+c$, and between $a-c$ and $b-c$. For example, the difference between the numbers 20 and 12 being 8, that difference will remain the same, whatever number we add to the numbers 20 and 12, and whatever numbers we subtract from them.

347. The proof is evident; for if $a-b=d$ we have also $(a+c)-(b+c)=d$; and also $(a-c)-(b-c)=d$.

348. *If we double the two numbers* a *and* b, *the difference will also become double.* Thus, when, $a-b=d$, we shall have, $2a-2b=2d$; and, *generally*, $na-nb=nd$, *whatever value we give to* n.

CHAPTER II.

Of Arithmetical Proportion.

349. When two arithmetical ratios, or relations, are equal, this equality is called an *arithmetical proportion.*

Thus, when $a-b=d$ and $p-q=d$, so that the difference is the same between the numbers p and q, as between the numbers a and b, we say that these four numbers form an arithmetical proportion; which we write thus, $a-b=p-q$, expressing clearly by this, that the difference between a and b is equal to the difference between p and q.

350. An arithmetical proportion consists therefore of four terms, which must be such, that if we subtract the second from the first, the remainder is the same as when we subtract the fourth from the third. Thus, the four numbers 12, 7, 9, 4, form an arithmetical proportion, because $12-7=9-4$.*

* To shew that these terms make such a proportion, some write them thus; 12 . . 7 : : 9 . . 4.

351. *When we have an arithmetical proportion, as* $a - b = p - q$, *we may make the second and third change places, writing* $a - p = b - q$; *and this equality will be no less true;* for, since $a - b = p - q$, add b to both sides, and we have $a = b + p - q$; then subtract p from both sides, and we have $a - p = b - q$.

In the same manner, as $12 - 7 = 9 - 4$, so also

$$12 - 9 = 7 - 4.$$

352. *We may, in every arithmetical proportion, put the second term also in the place of the first, if we make the same transposition of the third and fourth.* That is to say, if $a - b = p - q$, we have also $b - a = q - p$. For $b - a$ is the negative of $a - b$, and $q - p$ is also the negative of $p - q$. Thus, since $12 - 7 = 9 - 4$, we have also $7 - 12 = 4 - 9$.

353. *But the great property of every arithmetical proportion is this; that the sum of the second and third term is always equal to the sum of the first and fourth.* This property, which we must particularly consider, is expressed also by saying that, the sum of the *means* is equal to the sum of the *extremes*. Thus, since $12 - 7 = 9 - 4$, we have $7 + 9 = 12 + 4$; and the sum we find is 16 in both.

354. In order to demonstrate this principal property, let $a - b = p - q$; if we add to both $b + q$, we have $a + q = b + p$; that is, the sum of the first and fourth terms is equal to the sum of the second and third. *And conversely, if four numbers.* a, b, p, q, *are such, that the sum of the second and third is equal to the sum of the first and fourth,* that is, if $b + p = a + q$, we conclude, without a possibility of mistake, that *these numbers are in arithmetical proportion,* and that $a - b = p - q$. For, since

$$a + q = b + p,$$

if we subtract from both sides $b + q$, we obtain $a - b = p - q$.

Thus, the numbers 18, 13, 15, 10, being such, that the sum of the means ($13 + 15 = 28$,) is equal to the sum of the extremes ($18 + 10 = 28$,) it is certain, that they also form an arithmetical proportion; and. consequently, that

$$18 - 13 = 15 - 10.$$

355. It is easy, by means of this property, to resolve the following question. The three first terms of an arithmetical proportion being given to find the fourth? Let a, b, p, be the three

first terms, and let us express the fourth by q, which it is required to determine, then $a + q = b + p$; by subtracting a from both sides, we obtain $q = b + p - a$.

Thus, *the fourth term is found by adding together the second and third, and subtracting the first from that sum.* Suppose, for example, that 19, 28, 13. are the three first terms given, the sum of the second and third is $= 41$; take from it the first, which is 19, there remains 22 for the fourth term sought, and the arithmetical proportion will be represented by $19 - 28 = 13 - 22$, or, by $28 - 19 = 22 - 13$, or, lastly, by $28 - 22 = 19 - 13$.

356. *When in an arithmetical proportion, the second term is equal to the third, we have only three numbers;* the property of which is this, that the first, minus the second, is equal to the second, minus the third; or, that the difference between the first and the second number is equal to the difference between the second and the third. The three numbers, 19, 15, 11, are of this kind, since $19 - 15 = 15 - 11$.

357. Three *such numbers are said to form a continued arithmetical proportion,* which is sometimes written thus, 19 : 15 : 11. *Such proportions are also called* arithmetical progressions. *particularly if a greater number of terms follow each other according to the same law.*

An arithmetical progression may be either *increasing*, or *decreasing*. The former distinction is applied when the terms go on increasing, that is to say, when the second exceeds the first, and the third exceeds the second by the same quantity; as in the numbers 4, 7, 10. The decreasing progression is that, in which the terms go on always diminishing by the same quantity, such as the numbers 9, 5, 1.

358. Let us suppose the numbers a, b, c, to be in arithmetical progression; then $a - b = b - c$, whence it follows, from the equality between the sum of the extremes and that of the means, that $2b = a + c$; and if we subtract a from both, we have

$$c = 2b - a.$$

359. So that *when the two first terms* a, b, *of an arithmetical progression are given, the third is found by taking the first from twice the second.* Let 1 and 3 be the two first terms of an arithmetical progression, the third will be $= 2 \times 3 - 1 = 5$. And these three numbers 1, 3, 5 give the proportion $1 - 3 = 3 - 5$.

360. By following the same method, we may pursue the arithmetical progression as far as we please; *we have only to find the fourth by means of the second and third, in the same manner* as we determined the third by means of the first and second, and so on. Let a be the first term, and b the second, the third will be $= 2b - a$, the fourth $= 4b - 2a - b = 3b - 2a$, the fifth $6b - 4a - 2b + a = 4b - 3a$, the sixth $= 8b - 6a - 3b + 2a = 5b - 4a$, the seventh $= 10b - 8a - 4b + 3a = 6b - 5a$, &c.

CHAPTER III.

Of Arithmetical Progressions.

361. We have remarked already, that a series of numbers composed of any number of terms, which always increase, or decrease by the same quantity, is called an *arithmetical progression.*

Thus, the natural numbers written in their order, (as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c.) form an arithmetical progression, because they constantly increase by unity; and the series 25, 22, 19, 16, 13, 10, 7, 4, 1, &c. is also such a progression, since the numbers constantly decrease by 3.

362. The number, or quantity, by which the terms of an arithmetical progression become greater or less, is called the *difference.* So that when the first term and the difference are given, we may continue the arithmetical progression to any length.

For example, let the first term $= 2$, and the difference $= 3$, and we shall have the following increasing progression; 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, &c. in which each term is found, by adding the difference to the preceding term.

363. It is usual to write the natural numbers, 1, 2, 3, 4, 5, &c. above the terms of such an arithmetical progression, in order that we may immediately perceive the rank which any term holds in the progression. These numbers written above the

terms, may be called *indices*; and the above example is written as follows:

Indices,	1	2	3	4	5	6	7	8	9	10	
Arith. Prog.	2,	5,	8,	11,	14,	17,	20,	23,	26,	29,	&c.

where we see that 29 is the tenth term.

364. Let a be the first term, and d the difference, the arithmetical progression will go on in the following order:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ a, & a+d, & a+2d, & a+3d, & a+4d, & a+5d, & a+6d, & \&c. \end{matrix}$$

whence it appears, that any term of the progression might be easily found, without the necessity of finding all the preceding ones, by means only of the first term a and the difference d. For example, the tenth term will be $= a + 9\,d$, the hundredth term $= a + 99\,d$, and generally, the term n will be

$$= a + (n - 1)\,d.$$

365. When we stop at any point of the progression, it is of importance to attend to the first and the last term, since the index of the last will represent the number of terms. *If, therefore, the first term* = a, *the difference* = d, *and the number of terms* = n, *we shall have the last term* = a + (n — 1) d, *which is consequently found by multiplying the difference by the number of terms minus one, and adding the first term to that product.* Suppose, for example, in an arithmetical progression of a hundred terms, the first term is $= 4$, and the difference $= 3$; then the last term will be $= 99 \times 3 + 4 = 301$.

366. When we know the first term a and the last z, with the number of terms n, we can find the difference d. For, since the last term $z = a + (n - 1)\,d$, if we subtract a from both sides, we obtain $z - a = (n - 1)\,d$. So that by subtracting the first term from the last, we have the product of the difference multiplied by the number of terms *minus* 1. We have, therefore, only to divide $z - a$ by $n - 1$ to obtain the required value of the difference d, which will be $= \frac{z - a}{n - 1}$. This result furnishes the following rule: *Subtract the first term from the last, divide the remainder by the number of terms minus* 1, *and the quotient will be the difference:* by means of which we may write the whole progression.

367. Suppose, for example, that we have an arithmetical progression of nine terms, whose first is $= 2$, and last $= 26$, and that it is required to find the difference. We must subtract the first term, 2, from the last, 26, and divide the remainder, which is 24, by $9 - 1$, that is, by 8; the quotient 3 will be equal to the difference required, and the whole progression will be

1	2	3	4	5	6	7	8	9
2,	5,	8,	11,	14,	17,	20,	23,	26.

To give another example, let us suppose that the first term $= 1$, the last $= 2$, the number of terms $= 10$, and that the arithmetical progression, answering to these suppositions, is required; we shall immediately have for the difference $\frac{2-1}{10-1} = \frac{1}{9}$, and thence conclude that the progression is

1	2	3	4	5	6	7	8	9	10
1,	$1\frac{1}{9}$,	$1\frac{2}{9}$,	$1\frac{3}{9}$,	$1\frac{4}{9}$,	$1\frac{5}{9}$,	$1\frac{6}{9}$,	$1\frac{7}{9}$,	$\frac{8}{9}$.	2.

Another example. Let the first term $= 2\frac{1}{3}$, the last $= 12\frac{1}{2}$, and the number of terms $= 7$; the difference will be

$$\frac{12\frac{1}{2} - 2\frac{1}{3}}{7-1} = \frac{10\frac{1}{6}}{6} = \frac{61}{36} = 1\frac{25}{36},$$

and consequently the progression

1	2	3	4	5	6	7
$2\frac{1}{3}$,	$4\frac{1}{36}$,	$5\frac{13}{18}$,	$7\frac{5}{12}$,	$9\frac{1}{9}$,	$10\frac{29}{36}$,	$12\frac{1}{2}$.

268. If now the first term a, the last term z, and the difference d, are given, we may from them find the number of terms n. For since $z - a = (n-1)d$, by dividing the two sides by d, we have $\frac{z-a}{d} = n - 1$. Now, n being greater by 1 than $n - 1$, we have $n = \frac{z-a}{d} + 1$; consequently, *the number of terms is found by dividing the difference between the first and the last term, or* z — a, *by the difference of the progression, and adding unity to the quotient,* $\frac{z-a}{d}$.

For example, let the first term $= 4$, the last $= 100$, and the difference $= 12$, the number of terms will be $\frac{100-4}{12} + 1 = 9$; and these nine terms will be,

1	2	3	4	5	6	7	8	9
4,	16,	28,	40,	52,	64,	76,	88,	100.

If the first term $= 2$, the last $= 6$, and difference $= 1\frac{1}{3}$, the number of terms will be $\frac{4}{1\frac{1}{3}} + 1 = 4$; and these four terms will be,

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2, & 3\frac{1}{3}, & 4\frac{2}{3}, & 6. \end{array}$$

Again, let the first term $= 3\frac{1}{3}$, the last $= 7\frac{2}{3}$, and the difference $= 1\frac{4}{9}$, the number of terms will be $= \frac{7\frac{2}{3} - 3\frac{1}{3}}{1\frac{4}{9}} + 1 = 4$; which are,

$$3\frac{1}{3}, \quad 4\frac{7}{9}, \quad 6\frac{2}{9}, \quad 7\frac{2}{3}.$$

369. It must be observed, however, that as the number of terms is necessarily an integer, if we had not obtained such a number for n, in the examples of the preceding article, the questions would have been absurd.

Whenever we do not obtain an integral number for the value of $\frac{z - a}{d}$, it will be impossible to resolve the question; and consequently, in order that questions of this kind may be possible, $z - a$ must be divisible by d.

370. From what has been said, it may be concluded, that we have always four quantities, or things, to consider in arithmetical progression;

I. The first term a.

II. The last term z.

III. The difference d.

IV. The number of terms n.

And the relations of these quantities to each other are such, that if we know three of them, we are able to determine the fourth; for,

I. *If* a, d, *and* n *are known, we have* $z = a + (n - 1)d$.

II. *If* z, d, *and* n *are known, we have* $a = z - (n - 1)d$.

III. *If* a, z, *and* n *are known, we have* $d = \frac{z - a}{n - 1}$.

IV. *If* a, z, *and* d *are known, we have* $n = \frac{z - a}{d} + 1$.

CHAPTER IV.

Of the Summation of Arithmetical Progressions.

371. It is often necessary also to find the sum of an arithmetical progression. This might be done by adding all the terms together; but as the addition would be very tedious, when the progression consisted of a great number of terms, a rule has been devised, by which the sum may be more readily obtained.

372. We shall first consider a particular given progression, such that the first term = 2, the difference = 3, the last term = 29, and the number of terms = 10;

1	2	3	4	5	6	7	8	9	10
2,	5,	8,	11,	14,	17,	20,	23,	26,	29.

We see, in this progression, that the sum of the first and the last term = 31; the sum of the second and the last but one = 31; the sum of the third and the last but two = 31, and so on; and thence we conclude, that the sum of any two terms equally distant, the one from the first, and the other from the last term, is always equal to the sum of the first and the last term.

373. The reasons of this may be easily traced. For, if we suppose the first $= a$, the last $= z$, and the difference $= d$, the sum of the first and the last term is $= a + z$; and the second term being $= a + d$, and the last but one $= z - d$, the sum of these two terms is also $= a + z$. Further, the third term being $a + 2d$, and the last but two $= z - 2d$, it is evident that these two terms also, when added together, make $a + z$. The demonstration may be easily extended to all the rest.

374. To determine, therefore, the sum of the progression proposed let us write the same progression term by term, inverted, and add the corresponding terms together, as follows:

$$\begin{array}{r}
2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 \\
29 + 26 + 23 + 20 + 17 + 14 + 11 + 8 + 5 + 2. \\
\hline
31 + 31 + 31 + 31 + 31 + 31 + 31 + 31 + 31 + 31
\end{array}$$

This series of equal terms is evidently equal to twice the sum of the given progression; now the number of these equal terms is 10, as in the progression, and their sum, consequently, = 10

$\times\ 31 = 310$. So that, since this sum is twice the sum of the arithmetical progression, the sum required must be $= 155$.

375. If we proceed in the same manner, with respect to any arithmetical progression, the first term of which is $= a$, the last $= z$, and the number of terms $= n$; writing under the given progression the same progression inverted, and adding term to term, we shall have a series of n terms, each of which will be $= a + z$; the sum of this series will consequently be $= n\,(a + z)$, and it will be twice the sum of the proposed arithmetical progression; which therefore will be $= \frac{n\,(a + z)}{2}$.

376. This result furnishes an easy method of finding the sum of any arithmetical progression; and may be reduced to the following rule:

Multiply the sum of the first and the last term by the number of terms, and half the product will be the sum of the whole progression.

Or, which amounts to the same, multiply the sum of the first and the last term by half the number of terms.

Or, multiply half the sum of the first and the last term by the whole number of terms. Each of these enunciations of the rule will give the sum of the progression.

377. It may be proper to illustrate this rule by some examples.

First, let it be required to find the sum of the progression of the natural numbers, 1, 2, 3, &c. to 100. This will be, by the first rule, $= \frac{100 \times 101}{2} = 50 \times 101 = 5050$.

If it were required to tell how many strokes a clock strikes in twelve hours; we must add together the numbers 1, 2, 3, as far as 12; now this sum is found immediately $= \frac{12 \times 13}{2} = 6 \times 13 = 78$. If we wished to know the sum of the same progression continued to 1000, we should find it to be 500500; and the sum of this progression continued to 10000, would be 50005000.

378. *Another question.* A person buys a horse, on condition that for the first nail he shall pay 5 halfpence, for the second 8, for the third 11, and so on, always increasing 3 halfpence more

for each following one; the horse having 32 nails, it is required to tell how much he will cost the purchaser?

In this question, it is required to find the sum of an arithmetical progression, the first term of which is 5, the difference $= 3$, and the number of terms $= 32$. We must therefore begin by determining the last term; we find it (by the rule in articles 365 and 370) $= 5 + 31 \times 3 = 98$. After which the sum required is easily found $= \frac{103 \times 32}{2} = 103 \times 16$; whence we conclude that the horse costs 1648 halfpence, or 3*l.* 8*s.* 8*d.*

379. Generally, let the first term be $= a$, the difference $= d$, and the number of terms $= n$; and let it be required to find, by means of these data, the sum of the whole progression. As the last term must be $= a + (n - 1) d$, the sum of the first and last will be $= 2a + (n - 1) d$. Multiplying this sum by the number of terms n, we have $2na + n(n - 1) d$; the sum required therefore will be $= na + \frac{n(n-1)d}{2}$.

This formula, if applied to the preceding example, or to $a = 5$, $d = 3$, and $n = 32$, gives $5 \times 32 + \frac{32 \times 31 \times 3}{2} = 160 + 1488 = 1648$; the same sum that we obtained before.

380. If it be required to add together all the natural numbers from 1 to n, we have, for finding this sum, the first term $= 1$, the last term $= n$, and the number of terms $= n$; wherefore the sum required is $= \frac{nn + n}{2} = \frac{n(n+1)}{2}$.

If we make $n = 1766$, the sum of all the numbers, from 1 to 1766, will be $= 883 \times 1767 = 1560261$.

381. *Let the progression of uneven numbers be proposed*, 1, 3, 5, 7, &c. continued to n terms, and let the sum of it be required:

Here the first term is $= 1$, the difference $= 2$, the number of terms $= n$; the last term will therefore be $= 1 + (n - 1) 2 = 2n - 1$, and consequently the sum required $= nn$.

The whole therefore consists in multiplying the number of terms by itself. So that *whatever number of terms of this progression we add together, the sum will be always a square, namely, the square of the number of terms.* This we shall exemplify as follows;

Indices,	1	2	3	4	5	6	7	8	9	10	&c.
Progress,	1,	3,	5,	7,	9,	11,	13,	15,	17,	19,	&c.
Sum,	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	&c.

382. Let the first term be $= 1$, the difference $= 3$, and the number of terms $= n$; we shall have the progression 1, 4, 7, 10, &c. the last term of which will be $1 + (n - 1) 3 = 3n - 2$; wherefore the sum of the first and the last term $= 3n - 1$, and consequently, the sum of this progression $= \frac{n(3n - 1)}{2} = \frac{3nn - n}{2}$. If we suppose $n = 20$, the sum will be $= 10 \times 59 = 590$.

383. Again. let the first term $= 1$, the difference $= d$, and the number of terms $= n$; then the last term will be $= 1 + (n - 1) d$. Adding the first, we have $2 + (n - 1) d$, and multiplying by the number of terms, we have $2n + n(n - 1) d$; whence we deduce the sum of the progression $= n + \frac{n(n - 1)d}{2}$.

We subjoin the following small table:

If $d = 1$, the sum is $= n + \frac{n(n-1)}{2} = \frac{nn + n}{2}$

$d = 2$, $= n + \frac{2n(n-1)}{2} = nn$

$d = 3$, $= n + \frac{3n(n-1)}{2} = \frac{3nn - n}{2}$

$d = 4$, $= n + \frac{4n(n-1)}{2} = 2nn - n$

$d = 5$, $= n + \frac{5n(n-1)}{2} = \frac{5nn - 3n}{2}$

$d = 6$, $= n + \frac{6n(n-1)}{2} = 3nn - 2n$

$d = 7$, $= n + \frac{7n(n-1)}{2} = \frac{7nn - 5n}{2}$

$d = 8$, $= n + \frac{8n(n-1)}{2} = 4nn - 3n$

$d = 9$, $= n + \frac{9n(n-1)}{2} = \frac{9nn - 7n}{2}$

$d = 10$, $= n + \frac{10n(n-1)}{2} = 5nn - 4n$

CHAPTER V.

Of Geometrical Ratio.

384. THE *geometrical ratio* of two numbers is found by resolving the question, *how many times* is one of those numbers greater than the other? This is done by dividing one by the other; and the quotient, therefore, expresses the ratio required.

385. We have here three things to consider; 1st, the first of the two given numbers, which is called the *antecedent;* 2dly, the other number, which is called the *consequent;* 3dly, the ratio of the two numbers, or the quotient arising from the division of the antecedent by the consequent. For example, if the relation of the numbers 18 and 12 be required, 18 is the antecedent, 12 is the consequent, and the ratio will be $\frac{18}{12} = 1\frac{1}{2}$; whence we see, that the antecedent contains the consequent once and a half.

386. It is usual to represent geometrical relation by two points, placed one above the other, between the antecedent and the consequent. Thus $a : b$ means the geometrical relation of these two numbers, or the ratio of b to a.

We have already remarked, that this sign is employed to represent division, and for this reason we make use of it here; because, in order to know the ratio, we must divide a by b. The relation, expressed by this sign, is read simply, a is to b.

387. Relation therefore is expressed by a fraction, whose numerator is the antecedent, and whose denominator is the consequent. Perspicuity requires that this fraction should be always reduced to its lowest terms; which is done, as we have already shewn, by dividing both the numerator and denominator by their greatest common divisor. Thus, the fraction $\frac{18}{12}$ becomes $\frac{3}{2}$, by dividing both terms by 6.

388. So that relations only differ according as their ratios are different; and there are as many different kinds of geometrical relations as we can conceive different ratios.

The first kind is undoubtedly that in which the ratio becomes unity; this case happens when the two numbers are equal, as in $3 : 3$; $4 : 4$; $a : a$; the ratio is here 1, and for this reason we call it the relation of equality.

Next follow those relations in which the ratio is another whole number; in 4 : 2 the ratio is 2, and is called *double* ratio; in 12 : 4 the ratio is 3, and is called *triple* ratio; in 24 : 6 the ratio is 4, and is called *quadruple* ratio, &c.

We may next consider those relations whose ratios are expressed by fractions, as 12 : 9, where the ratio is $\frac{4}{3}$ or $1\frac{1}{3}$; 18 : 27, where the ratio is $\frac{2}{3}$, &c. We may also distinguish those relations in which the consequent contains exactly twice, thrice, &c. the antecedent; such are the relations 6 : 12, 5 : 15, &c. the ratio of which some call, *subduple, subtriple*, &c. ratios.

Further, we call that ratio *rational*, which is an expressible number; the antecedent and consequent being integers, as in 11 : 7, 8 : 15, &c. and we call that an *irrational* or *surd* ratio, which can neither be exactly expressed by integers, nor by fractions, as in $\sqrt{5} : 8$, $4 : \sqrt{3}$.

389. Let a be the antecedent, b the consequent, and d the ratio, we know already that a and b being given, we find $d = \frac{a}{b}$.

If the consequent b were given with the ratio, we should find the antecedent $a = bd$, because bd divided by b gives d. Lastly, when the antecedent a is given, and the ratio d, we find the consequent $b = \frac{a}{d}$; for, dividing the antecedent a by the consequent $\frac{a}{d}$, we obtain the quotient d, that is to say, the ratio.

390. Every relation $a : b$ remains the same, though we multiply. or divide the antecedent and consequent by the same number, because the ratio is the same. Let d be the ratio of $a : b$, we have $d = \frac{a}{b}$; now the ratio of the relation $na : nb$ is also $\frac{a}{b} = d$, and that of the relation $\frac{a}{n} : \frac{b}{n}$ is likewise $\frac{a}{b} = d$.

391. When a ratio has been reduced to its lowest terms, it is easy to perceive and enunciate the relation. For, example, when the ratio $\frac{a}{b}$ has been reduced to the fraction $\frac{p}{q}$, we say $a : b = p : q$, $a : b :: p : q$, which is read, a is to b as p is to q. Thus, the ratio of the relation 6 : 3 being $\frac{2}{1}$, or 2, we say 6 : 3 = 2 : 1.

We have likewise 18 : 12 = 3 : 2, and 24 : 18 = 4 : 3, and 30 : 45 = 2 : 3, &c. But if the ratio cannot be abridged, the relation will not become more evident; we do not simplify the relation by saying 9 : 7 = 9 : 7.

392. On the other hand, we may sometimes change the relation of two very great numbers into one that shall be more simple and evident, by reducing both to their lowest terms. For example, we can say 28844 : 14422 = 2 : 1; or,

10566 : 7044 = 3 : 2; or, 57600 : 25200 = 16 : 7.

393. In order, therefore, to express any relation in the clearest manner, it is necessary to reduce it to the smallest possible numbers. This is easily done, by dividing the two terms of the relation by their greatest common divisor. For example, to reduce the relation 57600 : 25200 to that of 16 : 7, we have only to perform the single operation of dividing the numbers 576 and 252 by 36, which is their greatest common divisor.

394. It is important, therefore, to know how to find the greatest common divisor of two given numbers; but this requires a rule, which we shall explain in the following chapter.

CHAPTER VI.

Of the greatest Common Divisor of two given numbers.

395. There are some numbers which have no other common divisor than unity, and when the numerator and denominator of a fraction are of this nature, it cannot be reduced to a more convenient form. The two numbers 48 and 35, for example, have no common divisor, though each has its own divisors. For this reason, we cannot express the relation 48 : 35 more simply, because the division of two numbers by 1 does not diminish them.

396. But when the two numbers have a common divisor, it is found by the following rule:

Divide the greater of the two numbers by the less; next, divide the preceding divisor by the remainder; what remains in this second division will afterwards become a divisor for a third division, in which the remainder of the preceding division will be the *dividend. We must continue this operation, till we arrive at a division that leaves no remainder; the divisor of this division, and consequently the last divisor, will be the greatest common divisor of the two given numbers.*

See this operation for the two numbers 576 and 252.

```
252) 576 (2
     504
     ---
      72) 252 (3
          216
          ---
           36) 72 (2
               72
               --
                0.
```

So that, in this instance, the greatest common divisor is 36.

397. It will be proper to illustrate this rule by some other examples. Let the greatest common divisor of the numbers 504 and 312 be required.

```
312) 504 (1
     312
     ---
     192) 312 (1
          192
          ---
          120) 192 (1
               120
               ---
                72) 120 (1
                     72
                    ---
                     48) 72 (1
                         48
                         --
                         24) 48 (2
                             48
                             --
                              0.
```

So that 24 is the greatest common divisor, and consequently the relation 504 : 312 is reduced to the form 21 : 13.

398. Let the relation 625 : 529 be given, and the greatest common divisor of these two numbers be required.

```
529) 625 (1
     529
     ---
      96) 529 (5
          480
          ---
           49) 96 (1
               49
               --
               47) 49 (1
                   47
                   --
                    2) 47 (23
                       46
                       --
                        1) 2 (2
                           2
                           -
                           0.
```

Wherefore 1 is, in this case, the greatest common divisor, and consequently we cannot express the relation 625 : 529 by less numbers, nor reduce it to less terms.

399. It may be proper, in this place, to give a demonstration of the rule. In order to this, let a be the greater and b the less of the given numbers; and let d be one of their common divisors; it is evident that a and b being divisible by d, we may also divide the quantities $a - b$, $a - 2b$, $a - 3b$, and, in general, $a - nb$ by d.

400. The converse is no less true; that is to say, if the numbers b and $a - nb$ are divisible by d, the number a will also be divisible by d. For nb being divisible by d, we could not divide $a - nb$ by d, if a were not also divisible by d.

401. We observe further, that if d be the *greatest* common divisor of two numbers, b and $a - nb$, it will also be the greatest common divisor of the two numbers a and b. Since, if a greater common divisor could be found than d, for these numbers, a and b, that number would also be a common divisor of b and $a - nb$; and, consequently, d would not be the greatest common divisor of these two numbers. Now we have supposed d the greatest divisor common to b and $a - nb$; wherefore d must also be the greatest common divisor of a and b.

402. These three things being laid down, let us divide, according to the rule, the greater number a by the less b; and let us suppose the quotient $= n$; the remainder will be $a - nb$, which must be less than b. Now this remainder $a - nb$ having the same greatest common divisor with b, as the given numbers a and b, we have only to repeat the division, dividing the preceding divisor b by the remainder $a - nb$; the new remainder, which we obtain, will still have, with the preceding divisor, the same greatest common divisor, and so on.

403. We proceed in the same manner, till we arrive at a division without a remainder; that is, in which the remainder is nothing. Let p be the last divisor, contained exactly a certain number of times in its dividend; this dividend will therefore be divisible by p, and will have the form mp; so that the numbers p, and mp, are both divisible by p; and it is certain, that they have no greater common divisor, because no number can actually be divided by a number greater than itself. Consequently, this last divisor is also the greatest common divisor of the given numbers a and b, and the rule, which we laid down, is demonstrated.

404. We may give another example of the same rule, requiring the greatest common divisor of the numbers 1728 and 2304. The operation is as follows:

```
1728) 2304 (1
      1728
      ----
       576) 1728 (3
            1728
            ----
              0.
```

From this it follows, that 576 is the greatest common divisor, and that the relation 1728 : 2304 is reduced to 3 : 4; that is to say, 1728 is to 2304 the same as 3 is to 4.

CHAPTER VII.

Of Geometrical Proportions.

405. Two geometrical relations are equal, when their ratios are equal. This equality of two relations is called a *geometrical proportion;* and we write for example, $a : b = c : d$, or $a : b :: c : d$, to indicate that the relation $a : b$ is equal to the relation $c : d$; but this is more simply expressed by saying, a is to b as c to d. The following is such a proportion, $8 : 4 = 12 : 6$; for the ratio of the relation $8 : 4$ is $\frac{2}{1}$, and this is also the ratio of the relation $12 : 6$.

406. So that $a : b = c : d$ being a geometrical proportion, the ratio must be the same on both sides, and $\frac{a}{b} = \frac{c}{d}$; and, reciprocally, if the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal, we have $a : b :: c : d$.

407. A geometrical proportion consists therefore of four terms, such, that the first, divided by the second, gives the same quotient as the third divided by the fourth. Hence we deduce an important property, common to all geometrical proportion, which is, that *the product of the first and the last term is always equal to the product of the second and third;* or, more simply, that *the product of the extremes is equal to the product of the means.*

408. In order to demonstrate this property, let us take the geometrical proportion $a : b = c : d$, so that $\frac{a}{b} = \frac{c}{d}$. If we multiply both these fractions by b, we obtain $a = \frac{bc}{d}$, and multiplying both sides further by d, we have $a\,d = b\,c$. Now $a\,d$ is the product of the extreme terms, $b\,c$ is that of the means, and these two products are found to be equal.

409. *Reciprocally, if the four numbers* a, b, c, d, *are such, that the product of the two extremes* a *and* d *is equal to the product of the two means* b *and* c, *we are certain that they form a geometrical proportion.* For since $a\,d = b\,c$, we have only to divide both sides by $b\,d$, which gives us $\frac{a\,d}{b\,d} = \frac{b\,c}{b\,d}$, or $\frac{a}{b} = \frac{c}{d}$, and consequently $a : b = c : d$.

410. *The four terms of a geometrical proportion, as* $a : b = c : d$, *may be transposed in different ways, without destroying the proportion. For the rule being always, that the product of the extremes is equal to the product of the means, or* $a d = b c$, *we may say*: 1st. $b : a = d : c$; 2dly. $a : c = b : d$; 3dly. $d : b = c : a$; 4thly. $d : c = b : a$.

411. Besides these four geometrical proportions, we may deduce some others from the same proportion, $a : b = c : d$. *We may say, the first term, plus the second, is to the first, as the third + the fourth is the third*; that is, $a + b : a = c + d : c$.

We may further say; *the first — the second is to the first as the third — the fourth is to the third*, or $a - b : a = c - d : c$.

For, if we take the product of the extremes and the means, we have $a c - b c = a c - a d$, which evidently leads to the equality $a d = b c$.

Lastly, it is easy to demonstrate, that $a + b : b = c + d : d$; and that $a - b : b = c - d : d$.

412. All the proportions which we have deduced from $a : b = c : d$, may be represented, generally, as follows:

$$m a + n b : p a + q b = m c + n d : p c + q d.$$

For the product of the extreme terms is $m p a c + n p b c + m q a d + n q b d$; which, since $a d = b c$, becomes $m p a c + n p b c + m q b c + n q b d$. Further, the product of the mean terms is $m p a c + m q b c + n p a d + n q b d$; or, since $a d = b c$, it is $m p a c + m q b c + n p b c + n q b d$; so that the two products are equal.

413. It is evident, therefore, that a geometrical proportion being given, for example, $6 : 3 = 10 : 5$, an infinite number of others may be deduced from it. We shall give only a few:

$$3 : 6 = 5 : 10;\ 6 : 10 = 3 : 5;\ 9 : 6 = 15 : 10;$$
$$3 : 3 = 5 : 5;\ 9 : 15 = 3 : 5;\ 9 : 3 = 15 : 5.$$

414. Since, in every geometrical proportion, the product of the extremes is equal to the product of the means, we may, when the three first terms are known, find the fourth from them. Let the three first terms be $24 : 15 = 40$ to .... as the product of the means is here 600, the fourth term multiplied by the first, that is by 24, must also make 600; consequently, by dividing 600 by 24, the quotient 25 will be the fourth term required, and the whole proportion will be $24 : 15 = 40 : 25$. In general,

therefore, if the three first terms are $a : b = c : \ldots.$ we put d for the unknown fourth letter; and since $a\,d = b\,c$, we divide both sides by a, and have $d = \frac{b\,c}{a}$. So that *the fourth term* is $= \frac{b\,c}{a}$, and *is found by multiplying the second term by the third, and dividing that product by the first term.*

415. This is the foundation of the celebrated *Rule of Three* in arithmetic; for what is required in that rule? We suppose three numbers given, and seek a fourth, which may be in geometrical proportion; so that the first may be to the second, as the third is to the fourth.

416. Some particular circumstances deserve attention here.

First, *if in two proportions the first and the third terms are the same,* as in $a : b = c : d$, and $a : f = c : g$, I say that *the two second and the two fourth terms will also be in geometrical proportion,* and that $b : d = f : g$. For, the first proportion being transformed into this, $a : c = b : d$, and the second into this, $a : c = f : g$, it follows that the relations $b : d$ and $f : g$ are equal, since each of them is equal to the relation $a : c$. For example, if $5 : 100 = 2 : 40$, and $5 : 15 = 2 : 6$, we must have $100 : 40 = 15 : 6$.

417. But if the two proportions are such, that the mean terms are the same in both, I say that the first terms will be in an inverse proportion to the fourth terms. That is to say, if $a : b = c : d$, and $f : b = c : g$, it follows that $a : f = g : d$. Let the proportions be, for example, $24 : 8 = 9 : 3$, and $6 : 8 = 9 : 12$, we have $24 : 6 = 12 : 3$. The reason is evident; the first proportion gives $a\,d = b\,c$; the second gives $f\,g = b\,c$; therefore, $a\,d = f\,g$, and $a : f = g : d$, or $a : g :: f : d$.

418. Two proportions being given, we may always produce a new one, by separately multiplying the first term of the one by the first term of the other, the second by the second, and so on, with respect to the other terms. Thus, the proportions $a : b = c : d$ and $e : f = g : h$ will furnish this, $a\,e : b\,f = c\,g : d\,h$. For the first giving $a\,d = b\,c$, and the second giving $e\,h = f\,g$, we have also $a\,d\,e\,h = b\,c\,f\,g$. Now $a\,d\,e\,h$ is the product of the extremes, and $b\,c\,f\,g$ is the product of the means in the new proportion; so that the two products being equal, the proportion is true.

419. Let the two proportions be, for example, $6 : 4 = 15 : 10$ and $9 : 12 = 15 : 20$, their combination will give the proportion $6 \times 9 : 4 \times 12 = 15 \times 15 : 10 \times 20$,

$$\text{or } 54 : 48 = 225 : 200,$$
$$\text{or } 9 : 8 = 9 : 8.$$

420. We shall observe lastly, that if two products are equal, $ad = bc$, we may reciprocally convert this equality into a geometrical proportion; for we shall always have one of the factors of the first product, in the same proportion to one of the factors of the second product, as the other factor of the second product is to the other factor of the first product; that is, in the present case, $a : c = b : d$, or $a : b = c : d$. Let $3 \times 8 = 4 \times 6$, and we may form from it this proportion, $8 : 4 = 6 : 3$, or this, $3 : 4 = 6 : 8$. Likewise, if $3 \times 5 = 1 \times 15$, we shall have

$$3 : 15 = 1 : 5, \text{ or } 5 : 1 = 15 : 3, \text{ or } 3 : 1 = 15 : 5.$$

CHAPTER VIII.

Observations on the Rules of Proportion and their utility.

421. This theory is so useful in the occurrences of common life, that scarcely any person can do without it. There is always a proportion between prices and commodities; and when different kinds of money are the subject of exchange, the whole consists in determining their mutual relations. The examples, furnished by these reflections, will be very proper for illustrating the principles of proportion, and shewing their utility by the application of them.

422. If we wished to know, for example, the relation between two kinds of money; suppose an old louis d'or and a ducat; we must first know the value of those pieces, when compared to others of the same kind. Thus, an old louis being, at Berlin, worth 5 rix dollars* and 8 drachms, and a ducat being worth 3 rix dollars, we may reduce these two values to one denomination; either to rix dollars, which gives the proportion 1 L : 1 D

* The rix dollar of Germany is valued at 92 cents 6 mills, and a drachm is one twenty-fourth part of a rix dollar.

$= 5\frac{1}{3}$ R : 3 R, or $= 16 : 9$; or to drachms, in which case we have 1 L : 1 D $= 128 : 72 = 16 : 9$. These proportions evidently give the true relation of the old louis to the ducat; for the equality of the products of the extremes and the means gives, in both, 9 louis $=$ 16 ducats; and, by means of this comparison, we may change any sum of old louis into ducats, and vice versa. Suppose it were required to tell how many ducats there are in 1000 old louis, we have this rule of three. If 9 louis are equal to 16 ducats, what are 1000 louis equal to? The answer will be $1777\frac{7}{9}$ ducats.

If, on the contrary, it were required to find how many old louis d'or there are in 1000 ducats, we have the following proportion. If 16 ducats are equal to 9 louis; what are 1000 ducats equal to? *Answer,* $562\frac{1}{2}$ old louis d'or.

423. Here, (at Petersburg,) the value of the ducat varies, and depends on the course of exchange. This course determines the value of the ruble in stivers, or Dutch half-pence, 105 of which make a ducat.

So that when the exchange is at 45 stivers, we have this proportion, 1 ruble : 1 ducat $= 45 : 105 = 3 : 7$; and hence this equality, 7 rubles $=$ 3 ducats.

By this we shall find the value of a ducat in rubles; for 3 ducats : 7 rubles $=$ 1 ducat : ..... *Answer,* $2\frac{1}{3}$ rubles.

If the exchange were at 50 stivers, we should have this proportion, 1 ruble : 1 ducat $= 50 : 105 = 10 : 21$, which would give 21 rubles $=$ 10 ducats; and we should have 1 ducat $= 2\frac{1}{10}$ rubles. Lastly, when the exchange is at 44 stivers, we have 1 ruble : 1 ducat $= 44 : 105$, and consequently 1 ducat $= 2\frac{17}{44}$ rubles $=$ 2 rubles $38\frac{7}{11}$ copecks.*

424. It follows from this, that we may also compare different kinds of money, which we have frequently occasion to do in bills of exchange. Suppose, for example, that a person of this place has 1000 rubles to be paid to him at Berlin, and that he wishes to known the value of this sum in ducats at Berlin.

The exchange is here at $47\frac{1}{2}$, that is to say, one ruble makes $47\frac{1}{2}$ stivers. In Holland, 20 stivers make a florin; $2\frac{1}{2}$ Dutch florins make a Dutch dollar. Further, the exchange of Holland

* A copeck is $\frac{1}{100}$ part of a ruble, as is easily deduced from the above.

with Berlin is at 142, that is to say, for 100 Dutch dollars, 142 dollars are paid at Berlin. Lastly, the ducat is worth 3 dollars at Berlin.

425. To resolve the questions proposed, let us proceed step by step. Beginning therefore with the stivers, since 1 ruble = $47\frac{1}{2}$ stivers, or 2 rubles = 95 stivers, we shall have 2 rubles : 95 stivers = 1000 : .... *Answer*, 47500 stivers. If we go further and say 20 stivers : 1 florin = 47500 stivers : .... we shall have 2375 florins. Further, $2\frac{1}{2}$ florins = 1 Dutch dollar, or 5 florins = 2 Dutch dollars; we shall therefore have 5 florins : 2 Dutch dollars = 2375 florins : .... *Answer*, 950 Dutch dollars.

Then taking the dollars of Berlin, according to the exchange at 142, we shall have 100 Dutch dollars : 142 dollars = 950 : the fourth term, 1349 dollars of Berlin. Let us, lastly, pass to the ducats, and say 3 dollars : 1 ducat = 1349 dollars : .... *Answer*, $449\frac{2}{3}$ ducats.

426. In order to render these calculations still more complete, let us suppose that the Berlin banker refuses, under some pretext or other, to pay this sum, and to accept the bill of exchange without five per cent. discount; that is, paying only 100 instead of 105. In that case, we must make use of the following proportion; 105 : 100 = $449\frac{2}{3}$: a fourth term, which is $428\frac{16}{63}$ ducats.

427. We have shewn that six operations are necessary, in making use of the Rule of Three; but we can greatly abridge those calculations, by a rule, which is called the *Rule of Reduction*. To explain this rule, we shall first consider the two antecedents of each of the six operations.

I.	2 rubles	:	95 stivers.
II.	20 stivers	:	1 Dutch flor.
III.	5 Dutch flor.	:	2 Dutch doll.
IV.	100 Dutch doll.	:	142 dollars.
V.	3 dollars	:	1 Ducat.
VI.	105 ducats	:	100 ducats.

If we now look over the preceding calculations, we shall observe, that we have always multiplied the given sum by the second terms, and that we have divided the products by the first; it is evident therefore, that we shall arrive at the same results, by multiplying, at once, the sum proposed by the product of all the second terms, and dividing by the product of all the first terms. Or, which amounts to the same thing, that we have only to make the following proportion; as the product of all the first terms is to the product of all the second terms, so is the given number of rubles to the number of ducats payable at Berlin.

428. This calculation is abridged still more, when amongst the first terms some are found that have common divisors with some of the second terms; for, in this case, we destroy those terms, and substitute the quotient arising from the division by that common divisor. The preceding example will, in this manner, assume the following form.*

Rubles $\cancel{2}$.	:	$19,\cancel{9}\cancel{5}$ stiv. 1000 rubles.
$\cancel{2}\cancel{0}$.	:	1 Dutch flor.
$\cancel{5}$.	:	$\cancel{2}$ Dutch dollars.
100.	:	142 dollars.
3.	:	1 ducat.
$\cancel{1}\cancel{0}\cancel{5}$,21.	:	$\cancel{5},\cancel{1}\cancel{0}\cancel{0}$ ducats.
$63\cancel{0}\cancel{0}$	:	$2698 = 10\cancel{0}\cancel{0}$:—

7) 26980.

9) 3854 (2

428 (2. *Answer,* $428\frac{16}{63}$ ducats.

429. The method, which must be observed, in using the rule of reduction, is this; we begin with the kind of money in question, and compare it with another, which is to begin the next relation, in which we compare this second kind with a third, and so on. Each relation, therefore, begins with the same kind, as the preceding relation ended with. This operation is continued, till we arrive at the kind of money which the answer requires; and, at the end, we reckon the fractional remainders.

* Divide the 1st and 9th by 2, the 3d and 12th by 20, the 5th and 12th (which is now 5) by 5, also the 2d and 11th by 5.

430. Other examples are added to facilitate the practice of this calculation.

If ducats gain at Hamburg 1 per cent. on two dollars banco; that is to say, if 50 ducats are worth, not 100, but 101 dollars banco; and if the exchange between Hamburg and Konigsberg is 119 drachms of Poland; that is, if 1 dollar banco gives 119 Polish drachms, how many Polish florins will 1000 ducats give?

30 Polish drachms make 1 Polish florin.

$$\begin{array}{rcll}
\text{Ducat } 1 & : & \cancel{2} & \text{doll. B°. } 1000 \text{ duc.} \\
\cancel{100}, 50 & : & 1\cancel{0}1 & \text{doll. B°.} \\
1 & : & 119 & \text{Pol. dr.} \\
30 & : & 1 & \text{Pol. flor.} \\
\hline
15\cancel{00} & : & 12019 = 10\cancel{00} & \text{duc.} : \ldots . \\
\hline
\end{array}$$

3) 120190.

5) 40063 (1.

8012 (3. *Answer*, 8012$\frac{2}{3}$ P. fl.

431. We may abridge a little further, by writing the number, which forms the third term, above the second row; for then the product of the second row, divided by the product of the first row, will give the answer sought.

Question. Ducats of Amsterdam are brought to Leipsick, having in the former city the value of 5 flor. 4 stivers current; that is to say, 1 ducat is worth 104 stivers, and 5 ducats are worth 26 Dutch florins. If, therefore, the *agio of the bank** at Amsterdam is 5 per cent., that is, if 105 currency are equal to 100 banco, and if the exchange from Leipsick to Amsterdam, in bank money, is 33$\frac{1}{4}$ per cent. that is, if for 100 dollars we pay at Leipsick 133$\frac{1}{4}$ dollars; lastly, 2 Dutch dollars making 5 Dutch florins; it is required to find how many dollars we must pay at Leipsick, according to these exchanges, for 1000 ducats?

* The difference of value between bank money and current money.

$\cancel{5}, \cancel{1000}$ ducats.

Ducats	$\cancel{5}$	:	26 flor. Dutch curr.
	$\cancel{105}$, 21	:	4, $\cancel{20}$, $\cancel{100}$ flor. Dutch banco.
	$\cancel{400}$, $\cancel{2}$	:	533 doll. of Leipsick.
	$\cancel{5}$	:	$\cancel{2}$ doll. banco.
	21	:	3) 55432 (1.
			7) 18477 (4.
			2639.

Answer, $2639\frac{13}{21}$ dollars, or 2639 dollars and 15 drachms.

CHAPTER IX.

Of Compound Relations.

432. COMPOUND RELATIONS are obtained, by multiplying the terms of two or more relations, the antecedents by the antecedents, and the consequents by the consequents; we say then, that the relation between those two products is *compounded* of the relations given.

Thus, the relations $a : b$, $c : d$, $e : f$, give the compound relation $ace : bdf$.*

433. A relation continuing always the same, when we divide both its terms by the same number, in order to abridge it, we may greatly facilitate the above composition by comparing the antecedents and the consequents, for the purpose of making such reductions as we performed in the last chapter.

For example, we find the compound relation of the following given relations, thus;

* Each of these three ratios is said to be one of the *roots* of the compound ratio.

Relations given.

12 : 25, 28 : 33, and 55 : 56.

$$\begin{array}{rcl}
\cancel{12}, \cancel{4}, 2 & : & 5, \cancel{25}. \\
\cancel{28} & : & \cancel{11}, \cancel{33}. \\
\cancel{55}, \cancel{11} & : & \cancel{7}, \cancel{56}. \\
\hline
2 & : & 5.
\end{array}$$

So that 2 : 5 is the compound relation required.

434. The same operation is to be performed, when it is required to calculate generally by letters; and the most remarkable case is that, in which each antecedent is equal to the consequent of the preceding relation. If the given relations are

$$\begin{array}{c}
a : b \\
b : c \\
c : d \\
d : e \\
e : a
\end{array}$$

the compound relation is 1 : 1.

435. The utility of these principles will be perceived, when it is observed, that the relation between two square fields is compounded of the relations of the lengths and the breadths.

Let the two fields, for example, be A and B; let A have 500 feet in length by 60 feet in breadth, and let the length of B be 360 feet, and its breadth 100 feet; the relation of the lengths will be 500 : 360, and that of the breadths 60 : 100. So that we have

$$\begin{array}{rcl}
\cancel{500}, 5 & : & 6, \cancel{360}. \\
\cancel{60} & : & \cancel{100}. \\
\hline
5 & : & 6
\end{array}$$

Wherefore the field A is to the field B, as 5 to 6.

436. *Another example.* Let the field A be 721 feet long, 88 feet broad; and let the field B be 660 feet long, and 90 feet broad; the relations will be compounded in the following manner.

$$\begin{array}{lrcr}
\text{Relation of the lengths,} & \cancel{720}, 8 & : & 15, \cancel{60}, \cancel{660} \\
\text{Relation of the breadths,} & \cancel{88}, \cancel{8}, 2 & : & \cancel{90} \\
\hline
\text{Relation of the fields A and B,} & 16 & : & 15.
\end{array}$$

437. Further, if it be required to compare two chambers with respect to the space, or contents, we observe that that relation is compounded of three relations; namely, of that of the lengths, that of the breadths, and that of the heights. Let there be, for example, the chamber A, whose length = 36 feet, breadth = 16 feet, and height = 14 feet, and the chamber B, whose length = 42 feet, breadth = 24 feet, and height = 10 feet; we shall have these three relations;

$$\begin{array}{lrcl}
\text{For the length} & \cancel{36}, \cancel{6} & : & \cancel{7}, \cancel{42}. \\
\text{For the breadth} & \cancel{16}, \cancel{4}, 2 & : & \cancel{6}, \cancel{24}. \\
\text{For the height} & \cancel{14}, 2 & : & 5, \cancel{10}. \\
\hline
 & 4 & : & 5
\end{array}$$

So that the contents of the chamber A : contents of the chamber B, as 4 : 5.

438. When the relations, which we compound in this manner, are equal, there result multiplicate relations. Namely, two equal relations give a *duplicate ratio* or *ratio of the squares;* three equal relations produce the *triplicate ratio* or *ratio of the cubes,* and so on, for example, the relations $a : b$ and $a : b$ give the compound relation $aa : bb$; wherefore we say, that the squares are in the duplicate ratio of their roots. And the ratio $a : b$ multiplied thrice, giving the ratio $a^3 : b^3$, we say that the cubes are in the triplicate ratio of their roots.

439. Geometry teaches, that two circular spaces are in the duplicate relation of their diameters; this means, that they are to each other as the squares of their diameters.

Let A be a circular space having the diameter = 45 feet, and B another circular space, whose diameter = 30 feet; the first space will be to the second, as 45×45 to 30×30; or, compounding these two equal relations,

$$\begin{array}{rcl}
\cancel{45}, \cancel{9}, 3 & : & 2, \cancel{6}, \cancel{30}. \\
\cancel{45}, \cancel{9}, 3 & : & 2, \cancel{6}, \cancel{30}. \\
\hline
9 & : & 4.
\end{array}$$

Wherefore the two areas are to each other as 9 to 4.

440. It is also demonstrated, that the solid contents of spheres are in the ratio of the cubes of the diameters. Thus, the diame-

ter of a globe A, being 1 foot, and the diameter of a globe B, being 2 feet, the solid contents of A will be to those of B, as $1^3 : 2^3$; or, as 1 to 8.

If therefore, the spheres are formed of the same substance, the sphere B will weigh 8 times as much as the sphere A.

441. It is evident, that we may, in this manner, find the weight of cannon balls, their diameters, and the weight of one, being given. For example, let there be the ball A, whose diameter $= 2$ inches, and weight $= 5$ pounds; and, if the weight of another ball be required, whose diameter is 8 inches, we have this proportion, $2^3 : 8^3 = 5$ to the fourth term, 320 pounds, which gives the weight of the ball B. For another ball C, whose diameter $= 15$ inches, we should have,

$$2^3 : 15^3 = 5 : \ldots . \textit{Answer}, \ 2109\tfrac{3}{8} \text{ lb.}$$

442. When the ratio of two fractions, as $\frac{a}{b} : \frac{c}{d}$, is required, we may always express it in integer numbers; for we have only to multiply the fractions by $b\,d$, in order to obtain the ratio $a\,d : b\,c$, which is equal to the other; from which results the proportion $\frac{a}{b} : \frac{c}{d} = a\,d : b\,c$. If, therefore, $a\,d$ and $b\,c$ have common divisors, the ratio may be reduced to less terms. Thus, $\frac{15}{24} : \frac{25}{36} = 15 \times 36 : 24 \times 25 = 9 : 10$.

443. If we wished to know the ratio of the fractions $\frac{1}{a}$ and $\frac{1}{b}$, it is evident, that we should have $\frac{1}{a} : \frac{1}{b} = b : a$; which is expressed by saying, that *two fractions, which have unity for their numerator, are in the reciprocal, or inverse ratio of their denominators. The same may be said of two fractions, which have any common numerator;* for $\frac{c}{a} : \frac{c}{b} = b : a$. *But if two fractions have their denominators equal,* as $\frac{a}{c} : \frac{b}{c}$, *they are in the direct ratio of the numerators;* namely, as $a : b$. Thus, $\frac{3}{8} : \frac{3}{16} = \frac{6}{16} : \frac{3}{16} = 6 : 3 = 2 : 1$, and $\frac{10}{7} : \frac{15}{7} = 10 : 15$, or, $= 2 : 3$.

444. It is observed, that in the free descent of bodies, a body

falls 16* feet in a second, that in two seconds of time it falls 64 feet, and that in three seconds it falls 144 feet; hence it is concluded, that the heights are to one another as the squares of the times; and that, reciprocally, the times are in the subduplicate ratio of the heights, or as the square roots of the heights.

If, therefore, it be required to find how long a stone must take to fall from the height of 2304 feet; we have $16 : 2304 = 1$ to the square of the time sought. So that the square of the time sought is 144; and, consequently, the time required is 12 seconds.

445. It is required to find how far, or through what height, a stone will pass, by descending for the space of an hour; that is, 3600 seconds. We say, therefore, as the squares of the times, that is, $1^2 : 3600^2$; so is the given height $= 16$ feet, to the height required.

$$1 : 12960000 = 16 : \ldots\ldots 207360000 \text{ height required.}$$

$$\begin{array}{r} 16 \\ \hline 77760000 \\ 1296 \\ \hline 207360000 \end{array}$$

If we now reckon 19200 feet for a league, we shall find this height to be 10800; and, consequently, nearly four times greater than the diameter of the earth.

446. It is the same with regard to the price of precious stones, which are not sold in the proportion of their weight; every body knows that their prices follow a much greater ratio. The rule for diamonds is, that the price is in the duplicate ratio of the weight, that is to say, the ratio of the prices is equal to the square of the ratio of the weights. The weight of diamonds is expressed in carats, and a carat is equivalent to 4 grains; if, therefore, a diamond of one carat is worth 10 livres, a diamond of 100 carats will be worth as many times 10 livres, as the square of 100 contains 1; so that we shall have, according to the rule of three,

* 15 is used in the original, as expressing the descent in Paris feet. It is here altered to English feet.

$1^2 \quad : \quad 100^2 = 10$ livres,

or $1 \quad : \quad 10000 \ = 10 : \ldots.$ *Answer*, 100000 livres.

There is a diamond in Portugal, which weighs 1680 carats; its price will be found, therefore, by making

$1^2 : \quad 1680^2 = 10$ liv $: \ldots.$ or

$1 \quad : 2822400 = 10 : \ 28224000$ liv.

447. The posts or mode of travelling in France furnish examples of compound ratios, as the price is according to the compound ratio of the number of horses, and the nnmber of leagues, or posts. For example, one horse costing 20 sous per post, it is required to find how much is to be paid for 28 horses and $4\frac{1}{2}$ posts.

We write first the ratio of horses,	1	:	28,
Under this ratio we put that of the stages or posts,	2	:	9,
And, compounding the two ratios, we have	2	:	252,

Or, $1 : 126 = 1$ livre to 126 francs or 42 crowns.

Another question. If I pay a ducat for eight horses, for 3 German miles, how much must I pay for thirty horses for four miles? The calculation is as follows:

$$\cancel{8}, \cancel{4} \ : \ 5, \cancel{15}, \cancel{30},$$
$$\cancel{3} \ : \ \cancel{4},$$

$1 \ : \ 5, = 1$ ducat : the 4th term, which will be 5 ducats.

448. The same composition occurs, when workmen are to be paid, since those payments generally follow the ratio compounded of the number of workmen, and that of the days which they have been employed.

If, for example, 25 sous per day be given to one mason, and it is required to find what must be paid to 24 masons who have worked for 50 days; we state this calculation;

$1 : 24$

$1 : 50$

$1 : 1200 = 25 : \ldots.$ 1500 francs.

25

20) 30000 (1500.

As, in such examples, five things are given, the rule, which serves to resolve them, is sometimes called, in books of arithmetic, The Rule of Five.

CHAPTER X.

Of Geometrical Progressions.

449. A SERIES of numbers, which are always becoming a certain number of times greater or less, is called a *geometrical progression*, because each term is constantly to the following one in the same geometrical ratio. And the number which expresses how many times each term is greater than the preceding, is called the *exponent*. Thus, when the first term is 1 and the exponent = 2, the geometrical progression becomes,

Terms	1	2	3	4	5	6	7	8	9	&c.
Prog.	1,	2,	4,	8,	16,	32,	64,	128,	256,	&c.

the numbers 1, 2, 3, &c. always marking the place which each term holds in the progression.

450. If we suppose, in general, the first term $= a$, and the exponent $= b$, we have the following geometrical progression;

$$\begin{array}{lllllllll} & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8 \ldots n \\ Prog. & a, & ab, & ab^2, & ab^3, & ab^4, & ab^5, & ab^6, & ab^7 \ldots ab^{n-1}. \end{array}$$

So that, when this progression consists of n terms, the last term is $= ab^{n-1}$. We must remark here, that if the exponent b be greater than unity, the terms increase continually; if the exponent $b = 1$, the terms are all equal; lastly, if the exponent b be less than 1, or a fraction, the terms continually decrease. Thus, when $a = 1$ and $b = \frac{1}{2}$, we have this geometrical progression;

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \tfrac{1}{32},\ \tfrac{1}{64},\ \tfrac{1}{128},\ \&c.$$

451. Here therefore we have to consider;

I. The first term, which we have called a.

II. The exponent, which we call b.

III. The number of terms, which we have expressed by n.

IV. The last term, which we have found $= ab^{n-1}$.

So that, when the three first of these are given, the last term is found, by multiplying the $n - 1$ power of b, or b^{n-1}, by the first term a.

If, therefore, the 50th term of the geometrical progression 1, 2, 4, 8, &c. were required, we should have $a = 1$, $b = 2$, and $n = 50$; consequently the 50th term $= 2^{49}$. Now 2^9 being $= 512$; 2^{10} will be $= 1024$. Wherefore the square of 2^{10}, or 2^{20}, $=$ 1048576, and the square of this number, or $1099511627776 = 2^{40}$. Multiplying therefore this value of 2^{40} by 2^9, or by 512, we have 2^{49} equal to 562949953421312.

452. One of the principal questions, which occurs on this subject, is to find the sum of all the terms of a geometrical progression; we shall therefore explain the method of doing this. Let there be given, first, the following progression, consisting of ten terms;

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 512,$$

the sum of which we shall represent by s, so that $s = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512$; doubling both sides, we shall have $2s = 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024$. Subtracting from this the progression represented by s, there remains $s = 1024 - 1 = 1023$; wherefore the sum required is 1023.

453. Suppose now, in the same progression, that the number of terms is undetermined and $= n$, so that the sum in question, or s, $= 1 + 2 + 2^2 + 2^3 + 2^4 \ldots\ldots 2^{n-1}$. If we multiply by 2, we have $2s = 2 + 2^2 + 2^3 + 2^4 \ldots\ldots 2^n$, and subtracting from this equation the preceding one, we have $s = 2^n - 1$. We see, therefore, that the sum required is found, by multiplying the last term, 2^{n-1}, by the exponent 2, in order to have 2^n, and subtracting unity from that product.

454. This is made still more evident by the following examples, in which we substitute successively, for n, the numbers 1, 2, 3, 4, &c.

$1 = 1$; $1 + 2 = 3$; $1 + 2 + 4 = 7$; $1 + 2 + 4 + 8 = 15$;
$1 + 2 + 4 + 8 + 16 = 31$; $1 + 2 + 4 + 8 + 16 + 32 = 63$, &c.

455. On this subject the following question is generally proposed. A man offers to sell his horse by the nails in his shoes, which are in number 32; he demands 1 liard for the first nail,

2 for the second, 4 for the third, 8 for the fourth, and so on, demanding for each nail twice the price of the preceding. It is required to find what would be the price of the horse?

This question is evidently reduced to finding the sum of all the terms of the geometrical progression 1, 2, 4, 8, 16, &c. continued to the 32d term. Now this last term is 2^{31}; and, as we have already found $2^{20} = 1048576$, and $2^{10} = 1024$, we shall have $2^{20} \times 2^{10} = 2^{30}$ equal to 1073741824; and multiplying again by 2, the last term $2^{31} = 2147483648$; doubling therefore this number, and subtracing unity from the product, the sum required becomes 4294967295 liards. These liards make $1073741823\frac{3}{4}$ sous, and dividing by 20, we have 53687091 livres, 3 sous, 9 deniers for the sum required.

456. Let the exponent now be = 3, and let it be required to find the sum of the geometrical progression 1, 3, 9, 27, 81, 243, 729, consisting of 7 terms. Suppose it $= s$, so that

$$s = 1 + 3 + 9 + 27 + 81 + 243 + 729;$$

we shall then have, multiplying by 3,

$$3s = 3 + 9 + 27 + 81 + 243 + 729 + 2187;$$

and subtracing the preceding series, we have $2s = 2187 - 1 = 2186$. So that the double of the sum is 2186, and consequently the sum required = 1093.

457. In the same progression, let the number of terms $= n$, and the sum $= s$; so that $s = 1 + 3 + 3^2 + 3^3 + 3^4 + \ldots . 3^{n-1}$. If we multiply by 3, we have $3s = 3 + 3^2 + 3^3 + 3^4 + \ldots . 3^n$. Subtracting from this the value of s, as all the terms of it, except the first, destroy all the terms of the value of $3s$, except the last, we shall have $2s = 3^n - 1$; therefore $s = \frac{3^n - 1}{2}$. So that the sum required is found by multiplying the last term by 3, subtracting 1 from the product, and dividing the remainder by 2. This will appear, also, from the following examples; $1 = 1$; $1 + 3 = \frac{3 \times 3 - 1}{2} = 4$; $1 + 3 + 9 = \frac{3 \times 9 - 1}{2} = 13$; $1 + 3 + 9 + 27 = \frac{3 \times 27 - 1}{2} = 40$; $1 + 3 + 9 + 27 + 81 = \frac{3 \times 81 - 1}{2} = 121$.

458. Let us now suppose, generally, the first term $= a$, the exponent $= b$, the number of terms $= n$, and their sum $= s$, so that

$$s = a = a b + a b^2 + a b^3 + a b^4 + \dots a b^{n-1}.$$

If we multiply by b, we have

$b s = ab + ab^2 + ab^3 + ab^4 + ab^5 + \dots a b^n$, and subtracting the above equation, there remains $(b - 1)\ s = a b^n - a$; whence we easily deduce the sum required $s = \dfrac{a b^n - a}{b - 1}$. *Consequently, the sum of any geometrical progression is found by multiplying the last term by the exponent of the progression, subtracting the first term from the product, and dividing the remainder by the exponent minus unity.*

459. Let there by a geometrical progression of seven terms, of which the first $= 3$; and let the exponent be $= 2$; we shall then have $a = 3$, $b = 2$, and $n = 7$; wherefore the last term $= 3 \times 2^6$, or $3 \times 64 = 192$; and the whole progression will be

$$3,\ 6,\ 12,\ 24,\ 48,\ 96,\ 192.$$

Further, if we multiply the last term 192 by the exponent 2, we have 384; subtracting the first term there remains 381; and dividing this by $b - 1$, or by 1, we have 381 for the sum of the whole progression.

460. Again, let there be a geometrical progression of six terms; let 4 be the first, and let the exponent be $= \frac{3}{2}$. The progression is

$$4,\ 6,\ 9,\ \tfrac{27}{2},\ \tfrac{81}{4},\ \tfrac{243}{8}.$$

If we multiply this last term $\frac{243}{8}$ by the exponent $\frac{3}{2}$, we shall have $\frac{729}{16}$; the subtraction of the first term 4 leaves the remainder $\frac{665}{16}$, which divided by $b - 1 = \frac{1}{2}$, gives $\frac{665}{8} = 83\frac{1}{8}$.

461. When the exponent is less than 1, and consequently, when the terms of the progression continually diminish, the sum of such a decreasing progression, which would go on to infinity, may be accurately expressed.

For example, let the first term $= 1$, the exponent $= \frac{1}{2}$, and the sum $= s$, so that

$$s = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \&c.$$

ad infinitum.

If we multiply by 2, we have

$$2s = 2 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \&c.$$

ad infinitum.

And, subtracting the preceding progression, there remains $s = 2$ for the sum of the proposed infinite progression.

462. If the first term $= 1$, the exponent $= \frac{1}{3}$, and the sum $= s$; so that

$s = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \&c.$ ad infinitum.

Multiplying the whole by 3, we have

$$3s = 3 + 1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \&c. \text{ ad infinitum};$$

and subtracting the value of s, there remains $2s = 3$; wherefore the sum $s = 1\frac{1}{2}$.

463. Let there be a progression, whose sum $= s$, first term $= 2$, and exponent $= \frac{3}{4}$; so that $s = 2 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \frac{81}{128} +$ &c. ad infinitum.

Multiplying by $\frac{4}{3}$, we have $\frac{4}{3}s = \frac{8}{3} + 2 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} + \frac{81}{12}$ + &c. ad infinitum. Subtracting now the progression s, there remains $\frac{1}{3}s = \frac{8}{3}$; wherefore the sum required $= 8$.

464. If we suppose, in general, the first term $= a$, and the exponent of the progession $= \dfrac{b}{c}$, so that this fraction may be less than 1, and consequently c greater than b; the sum of the progression carried on, ad infinitum, will be found thus;

Make $s = a + \dfrac{ab}{c} + \dfrac{ab^2}{cc} + \dfrac{ab^3}{3} + \dfrac{ab^4}{c^4} + \&c.$

Multiplying by $\dfrac{b}{c}$, we shall have

$$\frac{b}{c}s = \frac{ab}{c} + \frac{ab^2}{c^2} + \frac{ab^3}{c^3} + \frac{ab^4}{c^4} + \&c. \text{ ad infinitum.}$$

And, subtracting this equation from the preceding, there remains $\left(1 - \dfrac{b}{c}\right)s = a$.

Consequently $s = \dfrac{a}{1 - \dfrac{b}{c}}$.

If we multiply both terms of this fraction by c, we have $s = \dfrac{ac}{c-b}$. The sum of the infinite geometrical progression proposed is, therefore, found by dividing the first term a by 1 minus the exponent, or by multiplying the first term a by the denominator of the exponent, and dividing the product by the same denominator diminished by the numerator of the exponent.

465. In the same manner, we find the sums of progressions, the terms of which are alternately affected by the signs + and —. Let for example,

$$s = a - \frac{ab}{c} + \frac{ab^2}{c^2} - \frac{ab^3}{c^3} + \frac{ab^4}{c^4} - \&c.$$

Multiplying by $\frac{b}{c}$, we have

$$\frac{b}{c}s = \frac{ab}{c} - \frac{ab^2}{c^2} + \frac{ab^3}{c^3} - \frac{ab^4}{c^4} \&c.$$

And, adding to this equation to the preceding, we obtain $(1 + \frac{b}{c})\, s = a$. Whence we deduce the sum required $s = \dfrac{a}{1 + \frac{b}{c}}$, or $s = \dfrac{ac}{c + b}$.

466. We see, then, that if the first term $a = \frac{3}{5}$, and the exponent $= \frac{2}{5}$, that is to say, $b = 2$ and $c = 5$, we shall find the sum of the progression $\frac{3}{5} + \frac{6}{25} + \frac{12}{125} + \frac{24}{625} + \&c. = 1$; since, by subtracting the exponent from 1, there remains $\frac{3}{5}$. and by dividing the first term by that remainder, the quotient is 1.

Further, it is evident, if the terms be alternately positive and negative, and the progression assume this form;

$$\tfrac{3}{5} - \tfrac{6}{25} + \tfrac{12}{125} - \tfrac{24}{625} + \&c.$$

the sum will be

$$\frac{a}{1 + \frac{b}{c}} = \frac{\frac{3}{5}}{\frac{7}{5}} = \frac{3}{7}.$$

467. *Another example.* Let there be proposed the infinite progression,

$$\tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \tfrac{3}{10000} + \tfrac{3}{100000} + \&c.$$

The first term is here $\frac{3}{10}$, and the exponent is $\frac{1}{10}$. Subtracting this last from 1, there remains $\frac{9}{10}$, and, if we divide the first term by this fraction, we have $\frac{1}{3}$ for the sum of the given progression So that taking only one term of the progression, namely $\frac{3}{10}$, the error would be $\frac{1}{30}$.

Taking two terms $\frac{3}{10}+\frac{3}{100}=\frac{33}{100}$, there would still be wanting $\frac{1}{100}$ to make the sum $=\frac{1}{3}$.

468. *Another example.* Let there be given the infinite progression,

$$9+\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}+\&c.$$

The first term is 9, the exponent is $\frac{1}{10}$. So that 1, minus the exponent, $=\frac{9}{10}$; and $\frac{9}{\frac{9}{10}}=10$ the sum required.

This series is expressed by a decimal fraction, thus 9,9999999, &c.

CHAPTER XI.

Of Infinite Decimal Fractions.

469. It will be very necessary to shew how a vulgar fraction may be transformed into a decimal fraction; and, conversely, how we may express the value of a decimal fraction by a vulgar fraction.

470. *Let it be required, in general, to change the fraction* $\frac{a}{b}$, *into a decimal fraction; as this fraction expresses the quotient of the division of the numerator* a *by the denominator* b, *let us write, instead of* a, *the quantity* a,0000000, *whose value does not at all differ from that of* a, *since it contains neither tenth parts, nor hundredth parts, &c. Let us now divide this quantity by the number* b, *according to the common rules of division, observing to put the point in the proper place, which separates the decimal and the integers.* This is the whole operation, which we shall illustrate by some examples.

Let there be given first the fraction $\frac{1}{2}$, the division in decimals will assume this form,

$$\frac{2)\ 1,0000000}{0,5000000}=\frac{1}{2}.$$

Hence it appears, that $\frac{1}{2}$ is equal to 0,5000000 or to 0,5; which is sufficiently evident, since this decimal fraction represents $\frac{5}{10}$, which is equivalent to $\frac{1}{2}$.

471. Let $\frac{1}{3}$ be the given fraction, and we have,

$$\frac{3)\ 1,0000000}{0,3333333}\text{ \&c.} = \frac{1}{3}.$$

This shews, that the decimal fraction, whose value is $= \frac{1}{3}$, cannot, strictly, ever be discontinued, and that it goes on ad infinitum, repeating always the number 3. And, for this reason, it has been already shewn, that the fractions $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}$ &c. ad infinitum, added together, make $\frac{1}{3}$.

The decimal fraction, which expresses the value of $\frac{2}{3}$, is also continued ad infinitum; for we have,

$$\frac{3)\ 2,0000000}{0,6666666}\text{ \&c.} = \frac{2}{3}.$$

And besides, this is evident from what we have just said, because $\frac{2}{3}$ is the double of $\frac{1}{3}$.

472. If $\frac{1}{4}$ be the fraction proposed, we have

$$\frac{4)\ 1,0000000}{0,2500000}\text{ \&c.} = \frac{1}{4}.$$

So that $\frac{1}{4}$ is equal to 0,2500000, or to 0,25; and this is evident, since $\frac{2}{10} + \frac{5}{100} = \frac{25}{100} = \frac{1}{4}$.

In like manner, we should have for the fraction $\frac{3}{4}$,

$$\frac{4)\ 3,0000000}{0,7500000} = \frac{3}{4}.$$

So that $\frac{3}{4} = 0,75$; and in fact $\frac{7}{10} + \frac{5}{100} = \frac{75}{100} = \frac{3}{4}$.

The fraction $\frac{5}{4}$ is changed into a decimal fraction, by making

$$\frac{4)\ 5.0000000}{1,2500000} = \frac{5}{4}.$$

Now $1 + \frac{25}{100} = \frac{5}{4}$.

473. In the same manner, $\frac{1}{5}$ will be found equal to 0,2; $\frac{2}{5} = 0,4$; $\frac{3}{5} = 0,6$; $\frac{4}{5} = 0,8$; $\frac{5}{5} = 1$; $\frac{6}{5} = 1,2$, &c.

When the denominator is 6, we find $\frac{1}{6} = 0,1666666$, &c. which is equal to $0,666666 - 0,5$. Now $0,666666 = \frac{2}{3}$, and $0,5 = \frac{1}{2}$, wherefore $0,1666666 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

We find, also, $\frac{2}{6} = 0,333333$, &c. $= \frac{1}{3}$; but $\frac{3}{6}$ becomes $0,5000000 = \frac{1}{2}$. Further, $\frac{5}{6} = 0,833333 = 0,333333 + 0,5$, that is to say, $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

474. When the denominator is 7, the decimal fractions become more complicated. For example, we find $\frac{1}{7} = 0,142857$, however it must be observed, that these six figures are repeated

continually. To be convinced, therefore, that this decimal fraction precisely expresses the value of $\frac{1}{7}$, we may transform it into a geometrical progression, whose first term is $= \frac{142857}{1000000}$ and the exponent $= \frac{1}{1000000}$; and, consequently, the sum (art. 464) $= \frac{\frac{142857}{1000000}}{1 - \frac{1}{1000000}}$ (multiplying both terms by 1000000) $= \frac{142857}{999999} = \frac{1}{7}$.

475. We may prove, in a manner still more easy, that the decimal fraction which we have found is exactly $= \frac{1}{7}$; for substituting for its value the letter s, we have

$$\begin{aligned}
s &= 0{,}142857142857142857, \&c. \\
10\,s &= 1,\ 42857142857142857, \&c. \\
100\,s &= 14,\ 2857142857142857, \&c. \\
1000\,s &= 142,\ 857142857142857, \&c. \\
10000\,s &= 1428,\ 57142857142857, \&c. \\
100000\,s &= 14285,\ 7142857142857, \&c. \\
1000000\,s &= 142857,\ 142857142857, \&c. \\
\text{Subtract } s &= 0,\ 142857142857, \&c. \\
\hline
999999\,s &= 142857.
\end{aligned}$$

And, dividing by 999999, we have $s = \frac{142857}{999999} = \frac{1}{7}$. Wherefore the decimal fraction, which was made $= s$, is $= \frac{1}{7}$.

476. In the same manner $\frac{2}{7}$ may be transformed into a decimal fraction, which will be 0,28571428, &c. and this enables us to find more easily the value of the decimal fraction which we have supposed $= s$; because 0,28571428 &c. must be the double of it, and, consequently, $= 2\,s$. For we have seen that

$$\begin{aligned}
100\,s &= 14{,}28571428571 \ \&c. \\
\text{So that subtracting } 2\,s &= 0.28571428571 \ \&c. \\
\hline
\text{there remains } 98\,s &= 14 \\
\text{wherefore } s &= \tfrac{14}{98} = \tfrac{1}{7}.
\end{aligned}$$

We also find $\frac{3}{7} = 0{,}42857142857$ &c. which, according to our supposition, must be $= 3\,s$; now we have found that

$$\begin{aligned}
10\,s &= 1{,}42857142857 \ \&c. \\
\text{So that subtracting } 3\,s &= 0{,}42857142857 \ \&c. \\
\hline
\text{we have } 7\,s &= 1, \text{ wherefore } s = \tfrac{1}{7}.
\end{aligned}$$

477. When a proposed fraction, therefore, has the donominator 7, the decimal fraction is infinite, and 6 figures are continually repeated. The reason is, as it is easy to perceive, that when we continue the division we must return, sooner or later, to a remainder which we have had already. Now, in this division, 6 different numbers only can form the remainder, namely, 1, 2, 3, 4, 5, 6; so that, after the sixth division, at furthest, the same figures must return; but when the denominator is such as to lead to a division without remainder, these cases do not happen.

478. Suppose, now, that 8 is the denominator of the fraction proposed; we shall find the following decimal fractions; $\frac{1}{8} = 0.125$; $\frac{2}{8} = 0{,}25$; $\frac{3}{8} = 0{,}375$; $\frac{4}{8} = 0{,}5$; $\frac{5}{8} = 0{,}625$; $\frac{6}{8} = 0{,}75$; $\frac{7}{8} = 0{,}875$; &c.

If the denominator be 9, we have $\frac{1}{9} = 0{,}111$ &c. $\frac{2}{9} = 0{,}222$ &c. $\frac{3}{9} = 0{,}333$ &c.

If the denominator be 10, we $\frac{1}{10} = 0{,}1$; $\frac{2}{10} = 0{,}2$; $\frac{3}{10} = 0{,}3$. This is evident from the nature of the thing, as also that $\frac{1}{100} = 0{,}01$; that $\frac{37}{100} = 0{,}37$; that $\frac{256}{1000} = 0{,}256$; that $\frac{24}{10000} = 0{,}0024$ &c.

479. If 11 be the denominator of the given fraction, we shall have $\frac{1}{11} = 0{,}0909090$ &c. Now, suppose it were required to find the value of this decimal fraction; let us call it s, we shall have $s = 0{,}0909009$, and $10\ s = 00{,}909090$; further, $100\ s = 9{,}09090$. If, therefore, we subtract from the last the value of s, we shall have $99\ s = 9$, and consequently $s = \frac{9}{99} = \frac{1}{11}$. We shall have, also, $\frac{2}{11} = 0{,}181818$ &c.; $\frac{3}{11} = 0{,}272727$ &c.; $\frac{6}{11} = 0{,}545454$ &c.

480. There is a great number of decimal fractions, therefore, in which one, two, or more figures constantly recur, and which continue thus to infinity. Such fractions are curious, and we shall shew how their values may be easily found.

Let us first suppose, that a single figure is constantly repeated, and let us represent it by a, so that $s = 0{,}aaaaaaa$. We have

$$10\ s = a{,}aaaaaaa$$

and subtracting $$s = 0{,}aaaaaaa$$

we have $$9\ s = a;\ \text{wherefore}\ s = \frac{a}{9}.$$

When two figures are repeated, as $a\,b$, we have $s = 0,ababab a$. Therefore $100\,s = ab,ababab$; and if we subtract s from it, there remains $99\,s = a\,b$; consequently $s = \frac{a\,b}{94}$.

When three figures, as $a\,b\,c$, are found repeated, we have $s = 0,abcabcabc$; consequently, $1000\ s = abc,abcabc$; and subtract s from it, there remains $999\ s = a\,b\,c$; wherefore $s = \frac{a\,b\,c}{999}$, and so on.

Whenever, therefore, a decimal fraction of this kind occurs, it is easy to find its value. Let there be given, for example, 0,296296, its value will be $\frac{296}{999} = \frac{8}{37}$, dividing both terms by 27.

This fraction ought to give again the decimal fraction proposed; and we may easily be convinced that this is the real result, by dividing 8 by 9, and then that quotient by 3, because $27 = 3 \times 9$. We have

$$\begin{array}{r} 9)\ 8,0000000 \\ \hline 3)\ 0,8888888 \\ \hline 0,2962962, \&c. \end{array}$$

which is the decimal fraction that was proposed.

481. We shall give a curious example by changing the fraction $\frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$, into a decimal fraction. The operation is as follows.

$$\begin{array}{r} 2)\ 1,00000000000000 \\ \hline 3)\ 0,50000000000000 \\ \hline 4)\ 0,16666666666666 \\ \hline 5)\ 0,04166666666666 \\ \hline 6)\ 0,00833333333333 \\ \hline 7)\ 0,00138888888888 \end{array}$$

8) 0,00019841269841

9) 0,0000248015873O

10) 0,00000275573192

0,00000027557319.

SECTION IV.

OF ALGEBRAIC EQUATIONS, AND OF THE RESOLUTION OF THOSE EQUATIONS.

CHAPTER I.

Of the Solution of Problems in general.

ARTICLE 482.

The principal object of Algebra, as well as of all the parts of Mathematics, is to determine the value of quantities which were before unknown. This is obtained by considering attentively the conditions given, which are always expressed in known numbers. For this reason Algebra has been defined, *The science which teaches how to determine unknown quantities by means of known quantities.*

483. The definition, which we have now given, agrees with all that has been hitherto laid down. We have always seen the knowledge of certain quantities lead to that of other quantities, which before might have been considered as unknown.

Of this, addition will readily furnish an example. To find the sum of two or more given numbers, we had to seek for an unknown number which should be equal to those known numbers taken together.

In subtraction we sought for a number which should be equal to the difference of two known numbers.

A multitude of other examples are presented by multiplication, division, the involution of powers, and the extraction of roots. The question is always reduced to finding, by means of known quantities, another quantity till then unknown.

484. In the last section also, different questions were resolved, in which it was required to determine a number, that could not

be deduced from the knowledge of other given numbers, except under certain conditions.

All those questions were reduced to finding, by the aid of some given numbers, a new number which should have a certain connexion with them; and this connexion was determined by certain conditions, or properties, which were to agree with the quantity sought.

485. *When we have a question to resolve, we represent the number sought by one of the last letters of the alphabet, and then consider in what manner the given conditions can form an equality between two quantities.* This equality, which is represented by a kind of formula, called an *equation*, enables us at last to determine the value of the number sought, and consequently to resolve the question. Sometimes several numbers are sought; but they are found in the same manner by equations.

486. Let us endeavour to explain this further by an example. Suppose the following question, or *problem* was proposed.

Twenty persons, men and women, dine at a tavern; the share of the reckoning for one man is 8 sous,* that for one woman is 7 sous, and the whole reckoning amounts to 7 livres 5 sous; required, the number of men, and also of women?

In order to resolve this question, let us suppose that the number of men is $= x$; and now considering this number as known, we shall proceed in the same manner as if we wished to try whether it corresponded with the conditions of the question. Now, the number of men being $= x$, and the men and women making together twenty persons, it is easy to determine the number of the women, having only to subtract that of the men from 20, that is to say, the number of women $= 20 - x$.

But each man spends 8 sous; wherefore x men spend $8x$ sous.

And, since each woman spends 7 sous, $20 - x$ women must spend $140 - 7x$ sous.

So that adding together $8x$ and $140 - 7x$, we see that the whole 20 persons must spend $140 + x$ sous. Now, we know already how much they have spent; namely, 7 livres 5 sous, or 145 sous; there must be an equality therefore between 140

* A sous is $\frac{1}{20}$ of a livre; a livre $\frac{1}{6}$ of a crown, or 17 cents 6 mills.

$+x$ and 145; that is to say, we have the equation $140+x=145$, and thence we easily deduce $x=5$.

So that the company consisted of 5 men and 15 women.

487. *Another question of the same kind.*

Twenty persons, men and women, go to a tavern; the men spend 24 florins, and the women as much; but it is found that each man has spent 1 florin more than each woman. Required, the number of men and the number of women?

Let the number of men $=x$.

That of the women will be $=20-x$.

Now these x men having spent 24 florins, the share of each man is $\frac{24}{x}$ florins.

Further, the $20-x$ women having also spent 24 florins, the share of each woman is $\frac{24}{20-x}$ florins.

But we know that the share of each woman is one florin less than that of each man; if, therefore, we subtract 1 from the share of a man, we must obtain that of a woman; and consequently $\frac{24}{x}-1=\frac{24}{20-x}$. This, therefore, is the equation from which we are to deduce the value of x. This value is not found with the same ease as in the preceding question; but we shall soon see that $x=8$, which value corresponds to the equation; for $\frac{24}{8}-1=\frac{24}{12}$ includes the equality $2=2$.

488. It is evident how essential it is, in all problems, to consider the circumstances of the question attentively, in order to deduce from it an equation, that shall express by letters the numbers sought or unknown. After that, the whole art consists in resolving those equations, or deriving from them the values of the unknown numbers; and this shall be the subject of the present section.

489. We must remark, in the first place, the diversity which subsists among the questions themselves. In some, we seek only for one unknown quantiy; in others, we have to find two, or more; and it is to be observed, with regard to this last case, that in order to determine them all, we must deduce from the circumstances, or the conditions of the problem, as many equations as there are unknown quantities.

490. It must have already been perceived, that an equation consists of two parts separated by the sign of equality, =, to shew that those two quantities are equal to one another. We are often obliged to perform a great number of transformations on those two parts, in order to deduce from them the value of the unknown quantity; but these transformations must be all founded on the following principles; that *two quantities remain equal, whether we add to them, or subtract from them equal quantities; whether we multiply them, or divide them by the same number; whether we raise them both to the same power, or extract their roots of the same degree.*

491. The equations, which are resolved most easily, are those in which the unknown quantity does not exceed the first power, after the terms of the equation have been properly arranged; and we call them *simple equations,* or *equations of the first degree.* But if, after having reduced and ordered an equation, we find in it the square, or the second power of the unknown quantity, it may be called an *equation of the second degree,* which is more difficult to resolve.

CHAPTER II.

Of the Resolution of Simple Equations, or Equations of the first degree.

492. When the number sought, or the unknown quantity, is represented by the letter x, and the equation we have obtained is such, that one side contains only that x, and the other simply a known number, as for example, $x = 25$, the value of x is already found. We must always endeavour, therefore, to arrive at such a form, however complicated the equation may be when first formed. We shall give, in the course of this section, the rules which serve to facilitate these reductions.

493. Let us begin with the simplest cases, and suppose, first, that we have arrived at the equation $x + 9 = 16$; we see immediately that $x = 7$. And, in general, if we have found $x + a = b$, where a and b express any known numbers, we have only

to subtract a from both sides, to obtain the equation $x = b - a$, which indicates the value of x.

494. If the equation which we have found is $x - a = b$, we add a to both sides, and obtain the value of $x = b + a$.

We proceed in the same manner, if the equation has this form, $x - a = a\,a + 1$; for we shall have immediately $x = a\,a + a + 1$.

In this equation, $x - 8\,a = 20 - 6\,a$, we find $x = 20 - 6\,a + 8\,a$. or $x = 20 + 2\,a$.

And in this, $x + 6\,a = 20 + 3\,a$, we have $x = 20 + 3\,a - 6\,a$, or $x = 20 - 3\,a$.

495. If the original equation has this form, $x - a + b = c$, we may begin by adding a to both sides, which will give $x + b = c + a$; and then subtracting b from both sides, we shall find $x = c + a - b$. But we might also add $+ a - b$ at once to both sides; by this we obtain immediately $x = c + a - b$.

So in the following examples,

If $x - 2\,a + 3\,b = 0$, we have $x = 2\,a - 3\,b$.

If $x - 3\,a + 2\,b = 25 + a + 2\,b$, we have $x = 25 + 4\,a$.

If $x - 9 + 6\,a = 25 + 2\,a$, we have $x = 34 - 4\,a$.

496. When the equation which we have found has the form $a\,x = b$, we only divide the two sides by a, and we have $x = \frac{b}{a}$. But if the equation has the form $a\,x + b - c = d$, we must first make the terms that accompany $a\,x$ vanish, by adding to both sides $- b + c$; and then dividing the new equation, $a\,x = d - b + c$, by a, we shall have $x = \frac{d - b + c}{a}$.

We should have found the same value by subtracting $+ b - c$ from the given equation; that is, we should have had, in the same form, $a\,x = d - b + c$, and $x = \frac{d - b + c}{a}$. Hence,

If $2\,x + 5 = 17$, we have $2\,x = 12$, and $x = 6$.

If $3\,x - 8 = 7$, we have $3\,x = 15$, and $x = 5$.

If $4\,x - 5 - 3\,a = 15 + 9\,a$, we have $4\,x = 20 + 12\,a$, and, consequently, $x = 5 + 3\,a$.

497. When the first equation has the form $\frac{x}{a} = b$, we multiply both sides by a, in order to have $x = a\,b$.

But if we have $\frac{x}{a} + b - c = d$, we must first make $\frac{x}{a} = d - b + c$; after which, we find $x = (d - b + c)\,a = a\,d - a\,b + c$.

Let $\frac{1}{2}\,x - 3 = 4$, we have $\frac{1}{2}\,x = 7$, and $x = 14$.

Let $\frac{1}{3}\,x - 1 + 2\,a = 3 + a$, we have $\frac{1}{3}\,x = 4 - a$, and $x = 12 - 3\,a$.

Let $\frac{x}{a-1} - 1 = a$, we have $\frac{x}{a-1} = a + 1$, and $x = a\,a - 1$.

498. When we have arrived at such an equation as $\frac{a\,x}{b} = c$, we first multiply by b, in order to have $a\,x = b\,c$, and then dividing by a, we find $x = \frac{b\,c}{a}$.

If $\frac{a\,x}{b} - c = d$, we begin by giving the equation this form $\frac{a\,x}{b} = d + c$, after which we obtain the value of $a\,x = b\,d + b\,c$, and that of $x = \frac{b\,d + b\,c}{a}$.

Let us suppose $\frac{2}{3}\,x - 4 = 1$, we shall have $\frac{2}{3}\,x = 5$, and $2\,x = 15$; wherefore $x = \frac{15}{2}$, or $7\frac{1}{2}$.

If $\frac{3}{4}\,x + \frac{1}{2} = 5$, we have $\frac{3}{4}\,x = 5 - \frac{1}{2} = \frac{9}{2}$; wherefore $3\,x = 18$, and $x = 6$.

499. Let us now consider the case, which may frequently occur, in which two or more terms contain the letter x, either on one side of the equation or on both.

If those terms are all on the same side, as in the equation $x + \frac{1}{2}\,x + 5 = 11$, we have $x + \frac{1}{2}\,x = 6$, or $3x = 12$. and, lastly, $x = 4$.

Let $x + \frac{1}{2}\,x + \frac{1}{3}\,x = 44$, and let the value of x be required: if we first multiply by 3, we have $4\,x + \frac{3}{2}\,x = 132$; then multiplying by 2, we have $11\,x = 264$: wherefore $x = 24$. We might have proceeded more shortly, beginning with the reduction of the three terms which contain x, to the single term $\frac{11}{6}x$; and then dividing the equation $\frac{11}{6}x = 44$ by 11, we should have had $\frac{1}{6}\,x = 4$, wherefore $x = 24$.

Let $\frac{2}{3}\,x - \frac{3}{4}\,x + \frac{1}{2}\,x = 1$, we shall have, by reduction, $\frac{5}{12}\,x = 1$, and $x = 2\frac{2}{5}$.

Let, more generally, $ax - bx + cx = d$; this is the same as $(a-b+c)\,x = d$, whence we derive $x = \frac{d}{a-b+c}$.

500. When there are terms containing x on both sides of the equation, we begin by making such terms vanish from the side from which it is most easily done; that is to say, in which there are fewest of them.

If we have, for example, the equation $3x + 2 = x + 10$, we must first subtract x from both sides, which gives $2x + 2 = 10$; wherefore $2x = 8$, and $x = 4$.

Let $x + 4 = 20 - x$; it is evident that $2x + 4 = 20$; and consequently $2x = 16$, and $x = 8$.

Let $x + 8 = 32 - 3x$, we shall have $4x + 8 = 32$: then $4x = 24$, and $x = 6$.

Let $15 - x = 20 - 2x$, we shall have $15 + x = 20$, and $x = 5$.

Let $1 + x = 5 - \frac{1}{2}x$, we shall have $1 + \frac{3}{2}x = 5$; after that $\frac{3}{2}x = 4$; $3x = 8$; lastly, $x = \frac{8}{3} = 2\frac{2}{3}$.

If $\frac{1}{2} - \frac{1}{3}x = \frac{1}{3} - \frac{1}{4}x$, we must add $\frac{1}{3}x$, which gives $\frac{1}{2} = \frac{1}{3} + \frac{1}{12}x$; subtracting $\frac{1}{3}$, there remains $\frac{1}{12}x = \frac{1}{6}$, and multiplying by 12, we obtain $x = 2$.

If $1\frac{1}{2} - \frac{2}{3}x = \frac{1}{4} + \frac{1}{2}x$, we add $\frac{2}{3}x$, which gives $1\frac{1}{2} = \frac{1}{4} + \frac{7}{6}x$. Subtracting $\frac{1}{4}$, we have $\frac{7}{6}x = 1\frac{1}{4}$, whence we deduce $x = 1\frac{1}{14} = \frac{15}{14}$, by multiplying by 6, and dividing by 7.

501. If we have an equation, in which the unknown number x is a denominator, we must make the fraction vanish, by multiplying the whole equation by that denominator.

Suppose that we have found $\frac{100}{x} - 8 = 12$, we first add 8, and have $\frac{100}{x} = 20$; then multiplying by x, we have $100 = 20x$; and dividing by 20, we find $x = 5$.

Let $\frac{5x+3}{x-1} = 7$.

If we multiply by $x - 1$, we have $5x + 3 = 7x - 7$.

Subtracting $5x$, there remains $3 = 2x - 7$.

Adding 7, we have $2x = 10$. Wherefore $x = 5$.

502. Sometimes, also, radical signs are found in equations of the first degree. For example, a number x below 100 is required, and such, that the square root of $100 - x$ may be equal to 8, or $\sqrt{(100 - x)} = 8$; the square of both sides will be $100 - x = 64$, and adding x we have $100 = 64 + x$; whence we obtain $x = 100 - 64 = 36$.

Or, since $100 - x = 64$, we might have subtracted 100 from both sides; and we should then have had $-x = -36$; whence multiplying by -1, $x = 36$.

CHAPTER III.

Of the Solution of Questions relating to the preceding chapter.

503. *Question* I. To divide 7 into two such parts, that the greater may exceed the less by 3.

Let the greater part $= x$, the less will be $= 7 - x$; so that $x = 7 - x + 3$, or $x = 10 - x$; adding x, we have $2x = 10$; and, dividing by 2, the result is $x = 5$.

Answer. The greater part is therefore 5, and the less is 2.

Question II. It is required to divide a into two parts, so that the greater may exceed the less by b.

Let the greater part $= x$, the other will be $a - x$; so that $x = a - x + b$; adding x, we have $2x = a + b$; and dividing by 2, $x = \frac{a + b}{2}$.

Another Solution. Let the greater part $= x$; and, as it exceeds the less by b, it is evident that the less is smaller than the other by b, and therefore must be $= x - b$. Now these two parts, taken together, ought to make a; so that $2x - b = a$; adding b, we have $2x = a + b$, wherefore $x = \frac{a + b}{2}$, which is the value of the greater part; that of the less will be $\frac{a + b}{2} - b$, or $\frac{a + b}{2} - \frac{2b}{2}$, or $\frac{a - b}{2}$.

504. *Question* III. A father, who has three sons, leaves them 1600 crowns. The will specifies, that the eldest shall have 200

crowns more than the second, and that the second shall have 100 crowns more than the youngest. Required the share of each?

Let the share of the third son $= x$; then, that of the second will be $= x + 100$, and that of the first $= x + 300$. Now these three shares make up together 1600 crowns. We have, therefore.

$$3x + 400 = 1600$$
$$3x = 1200$$
$$\text{and } x = 400.$$

Answer. The share of the youngest is 400 crowns; that of the second is 500 crowns; and that of the eldest is 700 crowns.

505. *Question* IV. A father leaves four sons, and 8600 livres; according to the will, the share of the eldest is to be double that of the second, minus 100 livres; the second is to receive three times as much as the third, minus 200 livres; and the third is to recêive four times as much as the fourth, minus 300 livres. Required, the respective portions of these four sons.

Let us call x the portion of the youngest; that of the third son will be $= 4x - 300$; that of the second $= 12x - 1100$, and that of the eldest $= 24x - 2300$. The sum of these four shares must make 8600 livres. We have, therefore, the equation $41x - 3700 = 8600$, or $41x = 12300$, and $x = 300$.

Answer. The youngest must have 300 livres, the third son 900, the second 2500, and the eldest 4900.

506. *Question.* V. A man leaves 11000 crowns to be divided between his widow, two sons, and three daughters. He intends that the mother should receive twice the share of a son, and each son to receive twice as much as a daughter. Required, how much each of them is to receive?

Suppose the share of a daughter $= x$, that of a son is consequently $= 2x$, and that of the widow $= 4x$; the whole inheritance is therefore $3x + 4x + 4x$; so that $11x = 11000$, and $x = 1000$.

Answer. Each daughter receives	1000 crowns,
So that the three receive in all	3000
Each son receives 2000 crowns,	
So that both the sons receive	4000
And the mother receives	4000
Sum	11000 crowns.

507. *Question* VI. A father intends, by his will, that his three sons should share his property in the following manner; the eldest is to receive 1000 crowns less than half the whole fortune; the second is to receive 800 crowns less than the third of the whole property; and the third is to have 600 crowns less than the fourth of the property. Required, the sum of the whole fortune, and the portion of each son?

Let us express the fortune by x.

The share of the first son is	$\frac{1}{2}x - 1000$
That of the second	$\frac{1}{3}x - 800$
That of the third	$\frac{1}{4}x - 600$.

So that the three sons receive in all $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x - 2400$, and this sum must be equal to x.

We have, therefore, the equation $\frac{13}{12}x - 2400 = x$.

Subtrating x, there remains, $\frac{1}{12}x - 2400 = 0$.

Adding 2400, we have $\frac{1}{12}x = 2400$. Lastly multiplying by 12, the product is x equal to 28800.

Answer. The fortune consists of 28800 crowns, and

The eldest of the sons receives	13400 crowns
The second	8800
The youngest	6600
	28800 crowns.

508. *Question* VII. A father leaves four sons, who share his property in the following manner:

The first takes the half of the fortune, minus 3000 livres.

The second takes the third, minus 1000 livres.

The third takes exactly the fourth of the property.

The fourth takes 600 livres, and the fifth part of the property.

What was the whole fortune, and how much did each son receive?

Let the whole fortune be $= x$;

The eldest of the sons will have	$\frac{1}{2}x - 3000$
The second	$\frac{1}{3}x - 1000$
The third	$\frac{1}{4}x$
The youngest	$\frac{1}{5}x + 600$.

The four will have received in all $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x + \frac{1}{5}x - 3400$, which must be equal to x.

Whence results the equation $\frac{77}{60}x - 3400 = x$;

Subtracting x, we have $\frac{17}{60}x - 3400 = 0$;

Adding 3400, we have $\frac{17}{60}x = 3400$;

Dividing by 17, we have $\frac{1}{60}x = 200$;

Multiplying by 60, we have $x = 12000$.

Answer. The fortune consisted of 12000 livres.

The first son received	3000
The second	3000
The third	3000
The fourth	3000

509. *Question* VIII. To find a number such, that if we add to it its half, the sum exceeds 60 by as much as the number itself is less than 65.

Let the number $= x$, then $x + \frac{1}{2}x - 60 = 65 - x$; that is to say $\frac{3}{2}x - 60 = 65 - x$;

Adding x, we have $\frac{5}{2}x - 60 = 65$;

Adding 60, we have $\frac{5}{2}x = 125$;

Dividing by 5, we have $\frac{1}{2}x = 25$;

Multiplying by 2, we have $x = 50$.

Answer. The number sought is 50.

510. *Question* IX. To divide 32 into two such parts, that if the less be divided by 6, and the greater by 5, the two quotients taken together may make 6.

Let the less of the two parts sought $= x$; the greater will be $= 32 - x$; the first, divided by 6, gives $\frac{x}{6}$; the second, divided by 5, gives $\frac{32 - x}{5}$; now, $\frac{x}{6} + \frac{32 - x}{5} = 6$. So that multiplying by 5, we have $\frac{5}{6}x + 32 - x = 30$, or $-\frac{1}{6}x + 32 = 30$.

Adding $\frac{1}{6}x$, we have $32 = 30 + \frac{1}{6}x$.

Subtracting 30, there remains $2 = \frac{1}{6}x$.

Multiplying by 6, we have $x = 12$.

Answer. The two parts are; the less $= 12$, the greater $= 20$.

511. *Question* X. To find such a number that if multiplied by 5, the product shall be as much less than 40, as the number itself is less than 12.

Let us call this number x. It is less than 12 by $12 - x$. Taking the number x five times, we have $5x$, which is less than 40 by $40 - 5x$, and this quantity must be equal to $12 - x$.

We have therefore $40 - 5x = 12 - x$.

Adding $5x$, we have $40 = 12 + 4x$.

Subtracting 12, we have $28 = 4x$.

Dividing by 4, we have $x = 7$, the number sought.

512. *Question* XI. To divide 25 into two such parts, that the greater may contain the less 49 times.

Let the less part be $= x$, then the greater will be $= 25 - x$. The latter divided by the former ought to give the quotient 49; we have therefore $\frac{25 - x}{x} = 49$.

Multiplying by x, we have $25 - x = 49x$.

Adding x $\quad 25 = 50x$.

And dividing by 50 $\quad x = \frac{1}{2}$.

Answer. The less of the two numbers sought is $\frac{1}{2}$, and the greater is $24\frac{1}{2}$; dividing therefore the latter by $\frac{1}{2}$, or multiplying by 2, we obtain 49.

513. *Question* XII. To divide 48 into nine parts, so that every part may be always $\frac{1}{2}$ greater than the part which precedes it.

Let the first and least part $= x$, the second will be $= x + \frac{1}{2}$, the third $= x + 1$, &c.

Now these parts form an arithmetical progression, whose first term $= x$; therefore the ninth and last will be $= x + 4$. Adding those two terms together, we have $2x + 4$; multiplying this quantity by the number of terms, or by 9, we have $18x + 36$; and dividing this product by 2, we obtain the sum of all the nine parts $= 9x + 18$; which ought to be equal to 48. We have, therefore, $9x + 18 = 48$.

Subtracting 18, there remains $9x = 30$:

And dividing by 9, we have $\quad x = 3\frac{1}{3}$.

Answer. The first part is $3\frac{1}{3}$, and the nine parts succeed in the following order:

$$\begin{array}{ccccccccccccccccc} 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 \\ 3\frac{1}{3} & + & 3\frac{5}{6} & + & 4\frac{1}{3} & + & 4\frac{5}{6} & + & 5\frac{1}{3} & + & 5\frac{5}{6} & + & 6\frac{1}{3} & + & 6\frac{5}{6} & + & 7\frac{1}{3} \end{array}$$

which together make 48.

514. *Question* XIII. To find an arithmetical progression, whose first term $= 5$, last $= 10$, and sum $= 60$.

Here, we know neither the difference, nor the number of terms; but we know that the first and the last term would enable us to express the sum of the progression, provided only the number of terms was given. We shall, therefore, suppose this number $= x$, and express the sum of the progression by $\frac{15x}{2}$; now we know also that this sum is 60; so that $\frac{15x}{2} = 60$; $\frac{1}{2}x = 4$, and $x = 8$.

Now, since the number of terms is 8, if we suppose the difference $= z$, we have only to seek for the eighth term upon this supposition, and to make it $= 10$. The second term is $5 + z$, the third is $5 + 2z$, and the eighth is $5 + 7z$; so that

$$5 + 7z = 10$$
$$7z = 5$$
$$\text{and } z = \tfrac{5}{7}$$

Answer. The difference of the progression is $\frac{5}{7}$, and the number of terms is 8; consequently the progression is

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 + & 5\frac{5}{7} + & 6\frac{3}{7} + & 7\frac{1}{7} + & 7\frac{6}{7} + & 8\frac{4}{7} + & 9\frac{2}{7} + & 10 \end{array}$$

the sum of which $= 60$.

515. *Question* XIV. To find such a number, that if 1 be subtracted from its double, and the remainder be doubled, then if 2 be subtracted, and the remainder divided by 4, the number resulting from these operations shall be 1 less than the number sought.

Suppose this number $= x$; the double is $2x$; subtracting 1, there remains $2x - 1$; doubling this, we have $4x - 2$; subtracting 2, there remains $4x - 4$; dividing by 4, we have $x - 1$; and this must be one less than x; so that,

$$x - 1 = x - 1.$$

But this is what is called an *identical equation*; and shews that x is indeterminate; or that any number whatever may be substituted for it.

516. *Question* XV. I bought some ells of cloth at the rate of 7 crowns for 5 ells, which I sold again at the rate of 11 crowns for 7 ells, and I gained 100 crowns by the traffic. How much cloth was there?

Suppose that there were x ells of it; we must first see how much the cloth cost. This is found by the following proportion;

If five ells cost 7 crowns; what do x ells cost?

Answer, $\frac{7}{5}x$ crowns.

This was my expenditure. Let us now see my receipt: we must make the following proportion; as 7 ells are to 11 crowns, so are x ells to $\frac{11}{7}x$ crowns.

This receipt ought to exceed the expenditure by 100 crowns; we have, therefore, this equation.

$$\tfrac{11}{7}x = \tfrac{7}{5}x + 100;$$

Subtracting $\frac{7}{5}x$, there remains $\frac{6}{35}x = 100$.

Wherefore $6x = 3500$, and $x = 583\frac{1}{3}$.

Answer. There were $583\frac{1}{3}$ ells, which were bought for $816\frac{2}{3}$ crowns, and sold again for $916\frac{2}{3}$ crowns, by which means the profit was 100 crowns.

517. *Question* XVI. A person buys 12 pieces of cloth for 140 crowns. Two are white, three are black, and seven are blue. A piece of the black cloth costs two crowns more than a piece of the white, and a piece of blue cloth costs three crowns more than a piece of black. Required the price of each kind?

Let a white piece cost x crowns; then the two pieces of this kind will cost $2x$. Further, a black piece costing $x + 2$, the three pieces of this colour will cost $3x + 6$. Lastly, a blue piece costs $x + 5$; wherefore the seven blue pieces cost $7x + 35$. So that the twelve pieces amount in all to $12x + 41$.

Now, the actual and known price of these twelve pieces is 140 crowns; we have, therefore, $12x + 41 = 140$, and $12x = 99$; wherefore $x = 8\frac{1}{4}$;

So that a piece of white cloth costs $8\frac{1}{4}$ crowns; a piece of black cloth costs $10\frac{1}{4}$ crowns, and a piece of blue cloth costs $13\frac{1}{4}$ crowns.

518. *Question* XVII. A man, having bought some nutmegs, says that three nuts cost as much more than one sous as four cost him more than ten liards: Required, the price of those nuts?

We shall call x the excess of the price of three nuts above one sous, or four liards, and shall say; If three nuts cost $x + 4$ liards, four will cost, by the condition of the question, $x + 10$ liards. Now, the price of three nuts gives that of four nuts in another way also, namely, by the rule of three. We make $3 : x + 4 = 4$: *Answer,* $\dfrac{4x + 16}{3}$.

So that $\frac{4x + 16}{3} = x + 10$; or, $4x + 16 = 3x + 30$; wherefore $x + 16 = 30$.
and $x = 14$.

Answer. Three nuts cost 18 liards, and four cost 6 sous; wherefore each cost 6 liards.

519. *Question* XVIII. A certain person has two silver cups, and only one cover for both. The first cup weighs 12 ounces, and if the cover be put on it, it weighs twice as much as the other cup; but if the other cup be covered, it weighs three times as much as the first: Required, the weight of the second cup and that of the cover?

Suppose the weight of the cover $= x$ ounces; the first cup being covered will weigh $x + 12$ ounces. Now this weight being double that of the second cup, this cup must weigh $\frac{1}{2}x + 6$. If it be covered, it will weigh $\frac{3}{2}x + 6$; and this weight ought to be the triple of 12, that is, three times the weight of the first cup. We shall therefore have the equation $\frac{3}{2}x + 6 = 36$, or $\frac{3}{2}x = 30$; wherefore $\frac{1}{2}x = 10$ and $x = 20$.

Answer. The cover weighs 20 ounces, and the second cup weighs 16 ounces.

520. *Question* XIX. A banker has two kinds of change; there must be a pieces of the first to make a crown; and there must be b pieces of the second to make the same sum. A person wishes to have c pieces for a crown; how many pieces of each kind must the banker give him?

Suppose the banker gives x pieces of the first kind; it is evident that he will give $c - x$ pieces of the other kind. Now, the x pieces of the first are worth $\frac{x}{a}$ crown, by the proportion $a : 1 = x : \frac{x}{a}$; and the $c - x$ pieces of the second kind are worth $\frac{c - x}{b}$ crown, because we have $b : 1 = c - x : \frac{c - x}{b}$. So that $\frac{x}{a} + \frac{c - x}{b} = 1$; or $\frac{bx}{a} + c - x = b$; or $bx + ac - ax = ab$; or rather, $bx - ax = ab - ac$; whence we have $x = \frac{ab - ac}{b - a}$, or

$x = \frac{a(b-c)}{b-a}$. Consequently, $c - x = \frac{bc - ab}{b-a} = \frac{b(c-a)}{b-a}$.

Answer. The banker will give $\frac{a(b-c)}{b-a}$ pieces of the first kind, and $\frac{b(c-a)}{b-a}$ pieces of the second kind.

Remark. These two numbers are easily found by the rule of three, when it is required to apply the results which we have obtained. To find the first we say; $b - a : b - c = a : \frac{a(b-c)}{b-a}$. The second number is found thus; $b - a : c - a = b : \frac{b(c-a)}{b-a}$.

It ought to be observed also that a is less than b, and that c is also less than b; but at the same time greater than a, as the nature of the thing requires.

521. *Question* XX. A banker has two kinds of change; 10 pieces of one make a crown, and 20 pieces of the other make a crown. Now, a person wishes to change a crown into 17 pieces of money: How many of each must he have?

We have here $a = 10$, $b = 20$, and $c = 17$; which furnishes the following proportions;

I. $10 : 3 = 10 : 3$, so that the number of pieces of the first kind is 3.

II. $10 : 7 = 20 : 14$, and there are 14 pieces of the second kind.

522. *Question* XXI. A father leaves at his death several children, who share his property in the following manner;

The first receives a hundred crowns and the tenth part of the remainder.

The second receives two hundred crowns and the tenth part of what remains.

The third takes three hundred crowns and the tenth part of what remains.

The fourth takes four hundred crowns and the tenth part of what then remains, and so on.

Now it is found at the end, that the property has been divided equally among all the children. Required, how much it was, how many children there were, and how much each received?

This question is rather of a singular nature, and therefore deserves particular attention. In order to resolve it more easily, we shall suppose the whole fortune $= z$ crowns; and since all the children receive the same sum, let the share of each $= x$, by which means the number of children is expressed by $\frac{z}{x}$. This being laid down, we may proceed to the solution of the question, which will be as follows,

Sum, or property to be divided.	Order of the Children	Portion of each.	Differences.
z	1st.	$x = 100 + \frac{z-100}{10}$	
$z - x$	2d.	$x = 200 + \frac{z-x-200}{10}$	$100 - \frac{x-100}{10} = 0$
$z - 2x$	3d.	$x = 300 + \frac{z-2x-300}{10}$	$100 - \frac{x-100}{10} = 0$
$z - 3x$	4th.	$x = 400 + \frac{z-3x-400}{10}$	$100 - \frac{x-100}{10} = 0$
$z - 4x$	5th.	$x = 500 + \frac{z-4x-500}{10}$	$100 - \frac{x-100}{10} = 0$
$z - 5x$	6th.	$x = 600 + \frac{z-5x-600}{10}$	and so on.

We have inserted, in the last column, the differences which we obtain by subtracting each portion from that which follows. Now all the portions being equal, each of the differences must be $= 0$. And as it happens that all these differences are expressed exactly alike, it will be sufficient to make one of them equal to nothing, and we shall have the equation $100 - \frac{x-100}{10} = 0$. Multiplying by 10, we have $1000 - x - 100 = 0$, or $900 - x = 0$; consequently $x = 900$.

We now know, therefore, that the share of each child was 900 crowns; so that taking any one of the equations of the third column, the first for example, it becomes, by substituting the value of x, $900 = 100 + \frac{z-100}{10}$, whence we immediately obtain the value of z; for we have $9000 = 1000 + z - 100$, or $9000 = 900 + z$; wherefore $z = 8100$; and consequently $\frac{z}{x} = 9$.

Answer. So that the number of children $=9$; the fortune left by the father $=8100$ crowns; and the share of each child $=900$ crowns.

CHAPTER IV.

Of the Resolutions of two or more Equations of the First Degree.

523. IT frequently happens that we are obliged to introduce into algebraic calculations two or more unknown quantities, represented by the letters x, y, z; and if the question is determinate, we are brought to the same number of equations; from which, it is then required to deduce the unknown quantities. As we consider, at present, those equations only which contain no powers of an unknown quantity higher than the first, and no products of two, or more unknown quantities, it is evident that these equations will all have the form $az+by+cx=d$.

524. Beginning, therefore, with two equations, we shall endeavour to find from them the values of x and y. That we may consider this case in a general manner, let the two equations be, I. $ax+by=c$, and II. $fx+gy=h$, in which a, b, c, and f, g, h are known numbers. It is required, therefore, to obtain, from these two equations, the two unknown quantities x and y.

525. The most natural method of proceeding will readily present itself to the mind; which is to determine, from both equations, the value of one of the unknown quantities, x for example, and to consider the equality of those two values; for then we shall have an equation, in which the unknown quantity y will be found by itself, and may be determined by the rules which we have already given. Knowing y, we have only to substitute its value in one of the quantities that express x.

526. According to this rule, we obtain from the first equation, $x=\frac{c-by}{a}$, and from the second, $x=\frac{h-gy}{f}$; stating these two values equal to one another, we have this new equation;

$$\frac{c-by}{a}=\frac{h-gy}{f};$$

multiplying by a, the product is $c - by = \frac{ah - agy}{f}$; multiplying by f, the product is $fc - fby = ah - agy$; adding agy, we have $fc - fby + agy = ah$; subtracting fc, there remains $-fby + agy = ah - fc$; or $(ag - bf)y = ah - fc$; lastly, dividing by $ag - bf$, we have $y = \frac{ah - fc}{ag - bf}$.

In order now to substitute this value of y in one of the two values which we have found of x, as in the first, where $x = \frac{c - by}{a}$, we shall first have $-by = -\frac{abh + bcf}{ag - bf}$; whence $c - by = c - \frac{abh + bcf}{ag - bf}$, or $c - by = \frac{acg - bcf - abh + bcf}{ag - bf} = \frac{acg - abh}{ag - bf}$; and dividing by a, $x = \frac{c - by}{a} = \frac{cg - bh}{ag - bf}$.

527. *Question* I. To illustrate this method by examples let it be proposed to find two numbers, whose sum may be $= 15$, and difference $= 7$.

Let us call the greater number x, and the less y. We shall have,

$$\text{I. } x + y = 15, \text{ and II. } x - y = 7.$$

The first equation gives $x = 15 - y$, and the second $x = 7 + y$; whence results the new equation $15 - y = 7 + y$. So that $15 = 7 + 2y$; $2y = 8$, and $y = 4$; by which means we find $x = 11$.

Answer. The less number is 4, and the greater is 11.

528. *Question* II. We may also generalize the preceding question, by requiring two numbers, whose sum may be $= a$, and the difference $= b$.

Let the greater of the two be $= x$, and the less $= y$.

We shall have I. $x + y = a$, and II. $x - y = b$; the first equation gives $x = a - y$; and the second $x = b + y$.

Wherefore $a - y = b + y$; $a = b + 2y$; $2y = a - b$; lastly, $y = \frac{a - b}{2}$, and consequently $x = a - y = a - \frac{a - b}{2} = \frac{a + b}{2}$.

Answer. The greater number, or x, is $= \frac{a + b}{2}$, and the less, or y, is $= \frac{a - b}{2}$, or which comes to the same, $x = \frac{1}{2}a + \frac{1}{2}b$, and

$y = \frac{1}{2}a - \frac{1}{2}b$; and hence we derive the following theorem. *When the sum of any two numbers is* a, *and their difference is* b, *the greater of the two numbers will be equal to half the sum* plus *half the difference; and the less of the two numbers will be equal to half the sum* minus *half the difference.*

529. We may also resolve the same question in the following manner;

Since the two equations are $x + y = a$, and $x - y = b$; if we add one to the other, we have $2x = a + b$.

$$\text{Wherefore } x = \frac{a+b}{2}.$$

Lastly, subtracting the same equation from the other, we have $2y = a - b$; wherefore $y = \frac{a-b}{2}$.

530. *Question* III. A mule and an ass were carrying burdens amounting to some hundred weight. The ass complained of his, and said to the mule, I need only one hundred weight of your load, to make mine twice as heavy as yours. The mule answered, Yes, but if you gave me a hundred weight of yours, I should be loaded three times as much as you would be. How many hundred weight did each carry?

Suppose the mule's load to be x hundred weight, and that of the ass to be y hundred weight. If the mule gives one hundred weight to the ass, the one will have $y + 1$, and there will remain for the other $x - 1$; and since, in this case, the ass is loaded twice as much as the mule, we have $y + 1 = 2x - 2$.

Further, if the ass gives a hundred weight to the mule, the latter has $x + 1$, and the ass retains $y - 1$; but the burden of the former being now three times that of the latter, we have $x + 1 = 3y - 3$.

Our two equations will consequently be,

$$\text{I. } y + 1 = 2x - 2, \quad \text{II. } x + 1 = 3y - 3.$$

The first gives $x = \frac{y+3}{2}$, and the second gives $x = 3y - 4$; whence we have the new equation $\frac{y+3}{2} = 3y - 4$, which gives $y = \frac{11}{5}$, and also determines the value of x, which becomes $2\frac{3}{5}$.

Answer. The mule carried $2\frac{3}{5}$ hundred weight, and the ass carried $2\frac{1}{5}$ hundred weight.

531. When there are three unknown numbers, and as many equations; as, for example, I. $x+y-z=8$, II. $x+z-y=9$, III. $y+z-x=10$, we begin, as before, by deducing a value of x from each, and we have, from the Ist, $x=8+z-y$; from the IId, $x=9+y-z$; and from the IIId, $x=y+z-10$.

Comparing the first of these values with the second, and after that with the third also, we have the following equations;

I. $8+z-y=9+y-z$, II. $8+z-y=y+z-10$.

Now, the first gives $2z-2y=1$, and the second gives $2y=18$, or $y=9$; if therefore we substitute this value of y in $2z-2y=1$, we have $2z-18=1$, and $2z=19$, so that $z=9\frac{1}{2}$; it remains therefore only to determine x, which is easily found $=8\frac{1}{2}$.

Here it happens, that the letter z vanishes in the last equation, and that the value of y is found immediately. If this had not been the case, we should have had two equations between z and y, to be resolved by the preceding rule.

532. Suppose we had found the three following equations.

$$\text{I. } 3x+5y-4z=25, \quad \text{II. } 5x-2y+3z=46,$$
$$\text{III. } 3y+5z-x=62.$$

If we deduce from each the value of x, we shall have

$$\text{I. } x=\frac{25-5y+4z}{3}, \quad \text{II. } x=\frac{46+2y-3z}{5}$$
$$\text{III. } x=3y+5z-62.$$

Comparing these three values together, and first the third with the first, we have $3y+5z-62=\frac{25-5y+4z}{3}$; multiplying by 3, $9y+15z-186=25-5y+4z$; so that $9y+15z=211-5y+4z$, and $14y+11z=211$ by the first and the third. Comparing also the third with the second, we have $3y+5z-62=\frac{46+2y-3z}{5}$, or $46+2y-3z=15y+25z-310$, which when reduced is $356=13y+28z$.

We shall now deduce, from these two new equations, the value of y;

I. $211=14y+11z$; wherefore $14y=211-11z$, and $y=\frac{211-11z}{14}$.

II. $356 = 13y + 28z$; wherefore $13y = 356 - 28z$, and $y = \frac{356 - 28z}{13}$.

These two values form the new equation

$$\frac{211 - 11z}{14} = \frac{356 - 28z}{13},$$

which becomes, $2743 - 143z = 4984 - 392z$, or $249z = 2241$, whence $z = 9$.

This value being substituted in one of the two equations of y and z, we find $y = 8$; and lastly a similar substitution, in one of the three values of x, will give $x = 7$.

533. If there were more than three unknown quantities to be determined, and as many equations to be resolved, we should proceed in the same manner; but the calculations would often prove very tedious.

It is proper, therefore, to remark, that, in each particular case, means may always be discovered of greatly facilitating its resolution. These means consist in introducing into the calculation, beside the principal unknown quantities, a new unknown quantity arbitrarily assumed, such as, for example, the sum of all the rest; and when a person is a little practised in such calculations he easily perceives what is most proper to do. The following examples may serve to facilitate the application of these artifices.

534. *Question* IV. Three persons play together; in the first game, the first loses to each of the other two, as much money as each of them has. In the next, the second person loses to each of the other two, as much money as they have already. Lastly, in the third game, the first and the second person gain each, from the third, as much money as they had before. They then leave off, and find that they have all an equal sum, namely, 24 louis each. Required, with how much money each sat down to play?

Suppose that the stake of the first person was x louis, that of the second y, and that of the third z. Further, let us make the sum of all the stakes, or $x + y + z = s$. Now, the first person losing in the first game as much money as the other two have, he loses $s - x$; (for he himself having had x, the two others must have had $s - x$); wherefore there will remain to him $2x - s$; the second will have $2y$, and the third will have $2z$.

So that, after the first game, each will have as follows;

$$\text{the I. } 2x - s, \quad \text{the II. } 2y, \quad \text{the III. } 2z.$$

In the second game, the second person, who has now $2y$, loses as much money as the other two have, that is to say $s - 2y$; so that he has left $4y - s$. With regard to the others, they will each have double what they had; so that after the second game, the three persons have;

$$\text{the I. } 4x - 2s, \quad \text{the II. } 4y - s, \quad \text{the III. } 4z.$$

In the third game, the third person, who has now $4z$, is the loser; he loses to the first $4x - 2s$, and to the second $4y - s$; consequently after this game the three persons will have;

$$\text{the I. } 8x - 4s, \quad \text{the II. } 8y - 2s, \quad \text{the III. } 8z - s.$$

Now, each having at the end of this game 24 louis, we have three equations, the first of which immediately gives x, the second y, and the third z; further, s is known to be $= 72$, since the three persons have in all 72 louis at the end of the last game; but it is not necessary to attend to this at first. We have

I. $8x - 4s = 24$, or $8x = 24 + 4s$, or $x = 3 + \frac{1}{2}s$;

II. $8y - 2s = 24$, or $8y = 24 + 2s$, or $y = 3 + \frac{1}{4}s$;

III. $8z - s = 24$, or $8z = 24 + s$, or $z = 3 + \frac{1}{8}s$;

Adding these three values, we have

$$x + y + z = 9 + \tfrac{7}{8}s.$$

So that, since $x + y + z = s$, we have $s = 9 + \frac{7}{8}s$; wherefore $\frac{1}{8}s = 9$, and $s = 72$.

If we now substitute this value of s in the expressions which we have found for x, y, and z, we shall find that before they began to play, the first person had 39 louis; the second 21 louis; and the third 12 louis.

This solution shews, that by means of an expression for the sum of the three unknown quantities, we may overcome the difficulties which occur in the ordinary method.

535. Although the preceding question appears difficult at first, it may be resolved even without algebra. We have only to try to do it inversely. Since the players, when they left off, had each 24 louis, and, in the third game, the first and the second doubled the money, they must have had before that last game;

The I. 12, the II. 12, and the III. 48.

In the second game the first and the third doubled their money; so that before that game they had;

The I. 6, the II. 42, and the III. 24.

Lastly, in the first game, the second and the third gained each as much money as they began with; so that at first the three persons had;

I. 39, II. 21, III. 12.

The same result as we obtained by the former solution.

536. *Question* V. Two persons owe 29 pistoles; they have both money, but neither of them enough to enable him, singly to discharge this common debt: the first debtor says therefore to the second, if you give me $\frac{2}{3}$ of your money, I singly will immediately pay the debt. The second answers, that he also could discharge the debt, if the other would give him $\frac{3}{4}$ of his money. Required, how many pistoles each had?

Suppose that the first has x pistoles, and that the second has y pistoles.

We shall first have, $x + \frac{2}{3}y = 29$;

then also, $y + \frac{3}{4}x = 29$.

The first equation gives $x = 29 - \frac{2}{3}y$, and the second, $x = \frac{116 - 4y}{3}$; so that $29 - \frac{2}{3}y = \frac{116 - 4y}{3}$. From this equation, we get $y = 14\frac{1}{2}$; wherefore $x = 19\frac{1}{3}$.

Answer. The first debtor had $19\frac{1}{3}$ pistoles, and the second had $14\frac{1}{2}$ pistoles.

537. *Question* VI. Three brothers bought a vineyard for a hundred louis. The yougest says, that he could pay for it alone, if the second gave him half the money which he had; the second says, that if the eldest would give him only the third of his money, he could pay for the vineyard singly; lastly, the eldest asks only a fourth part of the money of the youngest, to pay for the vineyard himself. How much money had each?

Suppose the first had x louis; the second, y louis; the third, z louis; we shall then have the three following equations;

I. $x + \frac{1}{2}y = 100$. II. $y + \frac{1}{3}z = 100$.

III. $z + \frac{1}{4}x = 100$; two of which only give the value of x, namely, I. $x = 100 - \frac{1}{2}y$, III. $x = 400 - 4z$. So that we have the equation,

$100 - \frac{1}{2}y = 400 - 4z$, or $4z - \frac{1}{2}y = 300$, which must be combined with the second, in order to determine y and z. Now the second equation was $y + \frac{1}{3}z = 100$; we therefore deduce from it $y = 100 - \frac{1}{3}z$; and the equation found last being $4z - \frac{1}{2}y = 300$, we have $y = 8z - 600$. Consequently the final equation is,

$100 - \frac{1}{3}z = 8z - 600$; so that $8\frac{1}{3}z = 700$, or $\frac{25}{3}z = 700$, and $z = 84$. Wherefore $y = 100 - 28 = 72$, and $x = 64$.

Answer. The youngest had 64 louis, the second had 72 louis, and the eldest had 84 louis.

538. As, in this example, each equation contains only two unknown quantities, we may obtain the solution required in an easier way.

The first equation gives $y = 200 - 2x$; so that y is determined by x; and if we substitute this value in the second equation, we have $200 - 2x + \frac{1}{3}z = 100$; wherefore $\frac{1}{3}z = 2x - 100$, and $z = 6x - 300$.

So that z is also determined by x; and if we introduce this value into the third equation, we obtain $6x - 300 + \frac{1}{4}x = 100$, in which x stands alone, and which, when reduced to $25x - 1600 = 0$, gives $x = 64$. Consequently, $y = 200 - 128 = 72$, and $z = 384 - 300 = 84$.

539. We may follow the same method, when we have a greater number of equations. Suppose, for example, that we have in general;

$$\text{I. } u + \frac{x}{a} = n, \text{ II. } x + \frac{y}{b} = n, \text{ III. } y + \frac{z}{c} = n,$$

$$\text{IV. } z + \frac{u}{d} = n; \text{ or, reducing the fractions,}$$

$$\text{I. } au + x = an, \text{ II. } bx + y = bn, \text{ III. } cy + z = cn,$$
$$\text{IV. } dz + u = dn.$$

Here, the first equation gives immediately $x = an - au$, and, this value being substituted in the second, we have $abn - abu + y = bn$; so that $y = bn - abn + abu$; the substitution of this value, in the third equation, gives $bcn - abcn + abcu + z = cn$; wherefore $z = cn - bcn + abcn - abcu$; substituting this in the fourth equation, we have $cdn - bcdn + abcdn - abcdu + u = dn$. So that $dn - cdn + bcdn - abcdn = -abcdu + u$, or $(abcd - 1)u = abcdn - bcdn + cdn - dn$; whence we have $u = \frac{abcdn - bcdn + cdn - dn}{abcd - 1} = n \times \frac{(abcd - bcd + cd - d)}{abcd - 1}$.

Consequently, we shall have,

$$x = \frac{abcdn - acdn + adn - an}{abcd - 1} = n \times \frac{(abcd - acd + ad - a)}{abcd - 1}.$$

$$y = \frac{abcdn - abdn + abn - bn}{abcd - 1} = n \times \frac{(abcd - abd + ab - b)}{abcd - 1}.$$

$$z = \frac{abcdn - abcn + bcn - cn}{abcd - 1} = n \times \frac{(abcd - abc + bc - c)}{abcd - 1}.$$

$$u = \frac{abcdn - bcdn + cdn - dn}{abcd - 1} = n \times \frac{(abcd - bcd + cd - d)}{abcd - 1}.$$

540. *Question* VII. A captain has three companies, one of Swiss, another of Swabians, and a third of Saxons. He wishes to storm with part of these troops, and he promises a reward of 901 crowns, on the following condition;

That each soldier of the company, which assaults, shall receive 1 crown, and that the rest of the money shall be equally distributed among the two other companies.

Now it is found, that if the Swiss make the assault, each soldier of the other companies receives $\frac{1}{2}$ of a crown; that, if the Swabians assault, each of the others receives $\frac{1}{3}$ of a crown; lastly, that if the Saxons make the assault, each of the others receives $\frac{1}{4}$ of a crown. Required, the number of men in each company?

Let us suppose the number of Swiss $x =$, that of Swabians $= y$, and that of Saxons $= z$. And let us also make $x + y + z = s$, because it is easy to see, that by this, we abridge the calculation considerably. If, therefore, the Swiss make the assault, their number being x, that of the other will be $s - x$; now, the former receive 1 crown, and the latter half a crown; so that we shall have,

$$z + \tfrac{1}{2}s - \tfrac{1}{2}x = 901.$$

We find in the same manner, that if the Swabians make the assault, we have,

$$y + \tfrac{1}{3}s - \tfrac{1}{3}y = 901.$$

And lastly, that, if the Saxons mount the assault, we have,

$$z + \tfrac{1}{4}s - \tfrac{1}{4}z = 901.$$

Each of these three equtions will enable us to determine one of the unknown quantities x, y, z;

For the first gives $x = 1802 - s$,

the second gives $2y = 2703 - s$,

the third gives $3z = 3604 - s$,

If we now take the values of $6x$, $6y$, and $6z$, and write those values one above the other, we shall have,

$$6x = 10812 - 6s,$$
$$6y = 8109 - 3s,$$
$$6z = 7208 - 2s,$$

and adding; $6s = 26129 - 11s$, or $17s = 26129$; so that $s = 1537$; this is the whole number of soldiers, by which means we find,

$$x = 1802 - 1537 = 265;$$
$$2y = 2703 - 1537 = 1166, \text{ or } y = 583;$$
$$3z = 3604 - 1537 = 2067, \text{ or } z = 689.$$

Answer. The company of Swiss consists of 265 men; that of Swabians 583; and that of Saxons 689.

CHAPTER V.

Of the Resolution of Pure Quadratic Equations.

541. An *equation is said to be of the second degree, when it contains the square or the second power of the unknown quantity, without any of its higher powers.* An equation, containing likewise the third power of the unknown quantity, belongs to cubic equations, and its resolution requires particular rules. There are, therefore, only three kinds of terms in an equation of the second degree.

1. The terms in which the unknown quantity is not found at all, or which are composed only of known numbers.

2. The terms in which we find only the first power of the unknown quantity.

3. The terms which contain the square, or the second power of the unknown quantity.

So that x signifying an unknown quantity, and the letters a, b, c, d. &c. representing known numbers, the terms of the first kind will have the form a, the terms of the second kind will have the form $b\,x$, and the terms of the third kind will have the form $c\,x\,x$.

542. We have already seen, how two or more terms of the same kind may be united together, and considered as a single term.

For example, we may consider the formula $a\,x\,x - b\,x\,x + c\,x\,x$ as a single term, representing it thus $(a - b + c)\,x\,x$; since, in fact, $(a - b + c)$ is a known quantity.

And also, when such terms are found on both sides of the sign $=$, we have seen how they may be brought to one side, and then reduced to a single term. Let us take, for example, the equation,

$$2\,x\,x - 3\,x + 4 = 5\,x\,x - 8\,x + 11;$$

We first subtract $2\,x\,x$, and there remains

$$-3\,x + 4 = 3\,x\,x - 8\,x + 11;$$

then adding $8\,x$, we obtain,

$$5\,x + 4 = 3\,x\,x + 11;$$

Lastly, subtracting 11, there remains $3\,x\,x = 5\,x - 7$.

543. We may also bring all the terms to one side of the sign $=$, so as to leave only 0 on the other. It must be remembered, however, that when terms are transposed from one side to the other, their signs must be changed.*

Thus, the above equation will assume this form, $3\,x\,x - 5\,x + 7 = 0$; and, for this reason also, *the following general formula represents all equations of the second degree.*

$$a\,x\,x \pm b\,x \pm c = 0,$$

in which the sign $\pm$ is read *plus* or *minus*, and indicates that such terms may be sometimes positive and sometimes negative.

544. Whatever be the original form of a quadratic equation, it may always be reduced to this formula of three terms. If we have, for example, the equation

$$\frac{a\,x + b}{c\,x + d} = \frac{e\,x + f}{g\,x + h},$$

* That is, the quantity thus transposed is added to or subtracted from each side of the equation.

we must, first, reduce the fractions; multiplying, for this purpose, by $cx + d$, we have $ax + b = \frac{cexx + cfx + edx + fd}{gx + h}$, then by $gx + h$, we have $agxx + bgx + ahx + bh = cexx + cfx + edx + fd$, which is an equation of the second degree, and reducible to the three following terms, which we shall transpose by arranging them in the usual manner:

$$\begin{aligned} 0 = agxx &+ bgx + bh, \\ -cexx &+ ahx - fd, \\ &-cfx, \\ &-edx. \end{aligned}$$

We may exhibit this equation also in the following form, which is still more clear:

$$0 = (ag - ce)xx + (bg + ah - cf - ed)x + bh - fd.$$

545. Equations of the second degree, in which all the three kinds of terms are found, are called *complete*, and the resolution of them is attended with greater difficulties; for which reason we shall first consider those, in which one of the terms is wanting.

Now, if the term xx were not found in the equation, it would not be a quadratic, but would belong to those of which we have already treated. *If the term, which contains only known numbers, were wanting, the equation would have this form,* $\mathrm{a\,x\,x} \pm \mathrm{b\,x} = 0$, *which being divisible by* x, *may be reduced to* $\mathrm{a\,x} \pm \mathrm{b} = 0$, *which is likewise a simple equation, and belongs not to the present class.*

546. *But when the middle term, which contains the first power of* x, *is wanting, the equation assumes this form,* $\mathrm{a\,x\,x} \pm \mathrm{c} = 0$, *or* $\mathrm{a\,x\,x} = \mp \mathrm{c}$; as the sign of c may be either positive or negative.

We shall call such an equation a *pure* equation of the second degree, since the resolution of it is attended with no difficulty; for *we have only to divide by* a, *which gives* $\mathrm{x\,x} = \frac{\mathrm{c}}{\mathrm{a}}$; *and taking the square root of both sides, we find* $\mathrm{x} = \sqrt{\frac{\mathrm{c}}{\mathrm{a}}}$; *by means of which the equation is resolved.*

547. But there are three cases to be considered here. In the first, *when* $\frac{\mathrm{c}}{\mathrm{a}}$ *is a square number* (of which we can therefore really assign the root) *we obtain for the value of* x *a rational*

number, which may be either integer or fractional. For example, the equation $xx = 144$ gives $x = 12$. And $xx = \frac{9}{16}$ gives $x = \frac{3}{4}$.

The second variety is *when* $\frac{c}{a}$ *is not a square,* in which case *we must* therefore *be contented with the sign* $\surd$. If, for example, $xx = 12$, we have $x = \sqrt{12}$, the value of which may be determined by approximation, as we have already shown.

The third case is that in which $\frac{c}{a}$ *becomes a negative number; then the value of* x *is altogether impossible and imaginary; and this result proves that the question, which leads to such an equation, is in itself impossible.*

548. We shall also observe before proceeding further, that whenever it is required to extract the square root of a number, that root, as we have already remarked, has always two values, the one positive and the other negative. *Suppose we have the equation* $xx = 49$, *the value of* x *will be not only* $+7$, *but also* -7, *which is expressed by* $x = \pm 7$. So that all those questions admit of a double a answer; but it will be easily perceived that in several cases, in those, for example, which relate to a certain number of men, the negative value cannot exist.

549. In such equations, also, as $axx = bx$, where the known quantity c is wanting, there may be two values of x, though we find only one if we divide by x. In the equation $xx = 3x$, for example, in which it is required to assign such a value of x, that xx may become equal to $3x$, this is done by supposing $x = 3$, a value which is found by dividing the equation by x; but beside this value, there is also another, which is equally satisfactory, namely $x = 0$; for then $xx = 0$, and $3x = 0$. *Equations,* therefore, *of the second degree, in general, admit of two solutions, whilst simple equations admit only of one.*

We shall now illustrate, by some examples, what we have said with regard to pure equations of the second degree.

550. *Question* I. Required a number, the half of which multiplied by the third may produce 24.

Let this number $= x$; $\frac{1}{2}x$, multiplied by $\frac{1}{3}x$, must give 24; we shall therefore have the equation $\frac{1}{6}xx = 24$.

Multiplying by 6, we have $xx = 144$; and the extraction of the root gives $x = \pm 12$. We put $\pm$; for if $x = +12$, we have $\frac{1}{2}x = 6$, and $\frac{1}{3}x = 4$; now the product of these two numbers is 24; and if $x = -12$, we have $\frac{1}{2}x = -6$, and $\frac{1}{3}x = -4$, the product of which is likewise 24.

551. *Question* II. Required a number such, that by adding 5 to it, and subtracting 5 from it, the product of the sum by the difference would be 96.

Let this number be x, then $x + 5$, multiplied by $x - 5$, must give 96; whence results the equation, $xx - 25 = 96$.

Adding 25, we have $xx = 121$; and extracting the root, we have $x = 11$. Thus $x + 5 = 16$, also $x - 5 = 6$; and lastly, $6 \times 16 = 96$.

552. *Question* III. Required a number such, that by adding it to 10, and subtracting it from 10, the sum, multiplied by the remainder, or difference, will give 51.

Let x be this number; $10 + x$, multiplied by $10 - x$, must make 51, so that $100 - xx = 51$. Adding xx, and subtracting 51, we have $xx = 49$, the square root of which gives $x = 7$.

553. *Question* IV. Three persons, who had been playing, leave off; the first, with as many times 7 crowns, as the second has three crowns; and the second, with as many times 17 crowns, as the third has 5 crowns. Further, if we multiply the money of the first by the money of the second, and the money of the second by the money of the third, and lastly, the money of the third by that of the first, the sum of these three products will be $3830\frac{2}{3}$. How much money has each?

Suppose that the first player has x crowns; and since he has as many times 7 crowns, as the second has 3 crowns, we know that his money is to that of the second, in the ratio of 7 : 3.

We shall therefore make $7 : 3 = x$, to the money of the second player, which is therefore $\frac{3}{7}x$.

Further, as the money of the second player is to that of the third in the ratio of 17 : 5, we shall say, $17 : 5 = \frac{3}{7}x$ to the money of the third player, or to $\frac{15}{119}x$.

Multiplying x, or the money of the first player, by $\frac{3}{7}x$, the money of the second, we have the product $\frac{3}{7}xx$. Then $\frac{3}{7}x$, the money of the second, multiplyed by the money of the third, or by $\frac{15}{119}x$, gives $\frac{45}{833}xx$. Lastly, the money of the third, or $\frac{15}{119}x$ multiplied by x, or the money of the first, gives $\frac{15}{119}xx$. The sum of these three products is $\frac{3}{7}xx + \frac{45}{833}xx + \frac{15}{119}xx$; and, reducing these fractions to the same denominator, we find their sum $\frac{507}{833}xx$, which must be equal to the number $3830\frac{2}{3}$.

We have, therefore, $\frac{507}{833}xx = 3830\frac{2}{3}$.

So that $\frac{1521}{833}xx = 11492$, and $1521\,xx$ being equal to 9572836, dividing by 1521, we have $xx = \frac{9572836}{1521}$; and taking its root, we find $x = \frac{3094}{39}$. This fraction is reducible to lower terms if we divide by 13; so that $x = \frac{238}{3} = 79\frac{1}{3}$; and hence we conclude, that $\frac{3}{7}x = 34$, and $\frac{15}{119}x = 10$.

Answer. The first player has $79\frac{1}{3}$ crowns, the second has 34 crowns, and the third 10 crowns.

Remark. This calculation may be performed in an easier manner; namely, by taking the factors of the numbers which present themselves, and attending chiefly to the squares of those factors.

It is evident, that $507 = 3 \times 169$, and that 169 is the square of 13; then, that $833 = 7 \times 119$, and $119 = 7 \times 17$. Now we have $\frac{3 \times 169}{17 \times 49}xx = 3830\frac{2}{3}$, and if we multiply by 3, we have $\frac{9 \times 169}{17 \times 49}xx = 11492$. Let us resolve this number also into its factors; we readily perceive, that the first is 4, that is to say, that $11492 = 4 \times 2873$; further, 2873 is divisible by 17; so that $2873 = 17 \times 169$. Consequently our equation will assume the following form; $\frac{9 \times 169}{17 \times 49}xx = 4 \times 17 \times 169$, which, divided by 169, is reduced to $\frac{9}{17 \times 49}xx = 4 \times 17$; multiplying also by 17×49, and dividing by 9, we have $xx = \frac{4 \times 289 \times 49}{9}$, in which all the factors are squares; whence we have, without any further calculation, the root $x = \frac{2 \times 17 \times 7}{3} = \frac{238}{3} = 79\frac{1}{3}$, as before.

554. *Question* V. A company of merchants appoint a factor at Archangel. Each of them contributes for the trade, which they have in view, ten times as many crowns as there are part-

ners. The profit of the factor is fixed at twice as many crowns *per cent,* as there partners. Further, if we multiply the $\frac{1}{100}$ part of his total gain by $2\frac{2}{9}$, the number of partners will be found. Required, what is that number?

Let it be $= x$; and since, each partner has contributed $10\,x$, the whole capital is $= 10\,x\,x$. Now, for every hundred crowns, the factor gains $2\,x$, so that with the capital of $10\,x\,x$ his profit will be $\frac{1}{5}\,x^3$. The $\frac{1}{100}$ part of this gain is $\frac{1}{500}\,x^3$; multiplying by $2\frac{2}{9}$, or by $\frac{20}{9}$, we have $\frac{20}{4500}\,x^3$, or $\frac{1}{225}\,x^3$, and this must be equal to the number of partners, or x.

We have, therefore, the equation $\frac{1}{225}\,x^3 = x$, or $x^3 = 225\,x$; which appears, at first, to be of the third degree; but as we may divide by x, it is reduced to the quadratic $x\,x = 225$, whence $x = 15$.

Answer. There are fifteen partners, and each contributed 150 crowns.

CHAPTER VI.

Of the Resolution of Mixt Equations of the Second Degree.

555. An *equation of the second degree is said to be* mixt, *or complete,* when three kinds of terms are found in it, namely, that which contains the square of the unknown quantity, as* a x x; *that, in which the unknown quantity is found only of the first power, as* b x; *lastly, the kind of terms which is composed only of known quantities.* And since we may unite two or more terms of the same kind into one, and bring all the terms to one side of the sign =, the general form of a mixt equation of the second degree will be

$$a\,x\,x \mp b\,x \mp c = 0.$$

In this chapter, we shall show, how the value of x is derived from such equations. It will be seen that there are two methods of obtaining it.

556. An equation of the kind that we are now considering may be reduced, by division, to such a form, that the first term may contain only the square $x\,x$ of the unknown quantity x. We

* Sometimes called also affected.

shall leave the second term on the same side with x, and transpose the known term to the other side of the sign $=$. By these means our equation will assume the form $xx \pm px = \pm q$, in which p and q represent any known numbers, positive or negative; and the whole is at present reduced to determining the true value of x. We shall begin with remarking, that if $xx + px$ were a real square, the resolution would be attended with no difficulty, because it would only be required to take the square root of both sides.

557. But it is evident that $xx + px$ cannot be a square; since we have already seen, that *if a root consists of two terms, for example,* $x + n$, *its square always contains three terms, namely, twice the product of the two parts, besides the square of each part; that is to say, the square of* $x + n$ *is* $xx + 2nx + nn$. Now we have already on one side $xx + px$; *we may,* therefore, *consider* xx *as the square of the first part of the root, and in this case* px *must represent twice the product of* x, *the first part of the root, by the second part; consequently, this second part must be* $\frac{1}{2}p$, *and in fact the square of* $x + \frac{1}{2}p$, *is found to be* $xx + px + \frac{1}{4}pp$.

558. *Now* $xx + px + \frac{1}{4}pp$ *being a real square, which has for its root* $x + \frac{1}{2}p$, *if we resume our equation* $xx + px = q$, *we have only to add* $\frac{1}{4}pp$ *to both sides, which gives us* $xx + px + \frac{1}{4}pp = q + \frac{1}{4}pp$, *the first side being actually a square, and the other containing only known quantities. If, therefore, we take the square root of both sides, we find* $x + \frac{1}{2}p = \sqrt{(\frac{1}{4}pp + q)}$; *and subtracting* $\frac{1}{2}p$, *we obtain* $x = -\frac{1}{2}p + \sqrt{(\frac{1}{4}pp + q)}$; *and, as every square root may be taken either affirmatively or negatively, we shall have for* x *two values expressed thus;*

$$x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + q}.$$

559. This formula contains the rule by which all quadratic equations may be resolved, and it will be proper to commit it to memory, that it may not be necessary to repeat, every time, the whole operation which we have gone through. We may always arrange the equation, in such a manner, that the pure square xx may be found on one side, and the above equation have the form $xx + px = q$, where we see immediately that

$$x = -\tfrac{1}{2}p \pm \sqrt{\frac{1}{4}pp + q}.$$

560. The general rule, therefore, which we deduce from this, in order to resolve the equation $xx = -px + q$, is founded on this consideration :

That the unknown quantity x is equal to half the coefficient, or multiplier of x on the other side of the equation, *plus* or *minus* the square root of the square of this number, and the known quantity which forms the third term of the equation.

Thus if we had the equation $xx = 6x + 7$, we should immediately say, that $x = 3 \pm \sqrt{9+7} = 3 \pm 4$, whence we have these two values of x, I. $x = 7$; II. $x = -1$. In the same manner, the equation $xx = 10x - 9$, would give $x = 5 \pm \sqrt{25-9} = 5 \pm 4$, that is to say, the two values of x are 9 and 1.

561. This rule will be still better understood, by distinguishing the following cases. I. when p is an even number; II. when p is an odd number; and III. when p is a fractional number.

I. Let p be an even number, and the equation such, that $xx = 2px + q$; we shall, in this case, have $x = p \pm \sqrt{pp+q}$.

II. Let p be an odd number, and the equation $xx = px + q$; we shall here have $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + q}$; and since $\frac{1}{4}pp + q = \frac{pp+4q}{4}$, we may extract the square root of the denominator, and write $x = \frac{1}{2}p \pm \frac{\sqrt{pp+4q}}{2} = \frac{p \pm \sqrt{pp+4q}}{2}$.

III. Lastly, if p be a fraction, the equation may be resolved in the following manner; let the equation be $axx = bx + c$, or $xx = \frac{bx}{a} + \frac{c}{a}$, and we shall have by the rule, $x = \frac{b}{2a} \pm \sqrt{\frac{bb}{4aa} + \frac{c}{a}}$. Now, $\frac{bb}{4aa} + \frac{c}{a} = \frac{bb+4ac}{4aa}$, the denominator of which is a square; so that $x = \frac{b \pm \sqrt{bb+4ac}}{2a}$.

562. The other method of resolving mixt quadratic equations, is to transform them into pure equations. This is done by substitution; for example, in the equation $xx = px + q$, instead of the unknown quantity x, we may write another unknown quantity y, such, that $x = y + \frac{1}{2}p$; by which means, when we have determined y, we may immediately find the value of x.

If we make this substitution of $y + \frac{1}{2}p$ instead of x, we have $xx = yy + py + \frac{1}{4}pp$, and $px = py + \frac{1}{2}pp$; consequently our equation will become $yy + py + \frac{1}{4}pp = py + \frac{1}{2}pp + q$, which is first reduced, by subtracting py, to $yy + \frac{1}{4}pp = \frac{1}{2}pp + q$; and then, by subtracting $\frac{1}{4}pp$, to $yy = \frac{1}{4}pp + q$. This is a pure quadratic equation, which immediately gives $y = \pm\sqrt{\frac{1}{4}pp + q}$. Now, since $x = y + \frac{1}{2}p$, we have $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + q}$, as we found it before. We have only, therefore, to illustrate this rule by some examples.

563. *Question* I. There are two numbers; one exceeds the other by 6, and their product is 91. What are those numbers?

If the less is x, the other is $x + 6$, and their product $xx + 6x = 91$. Subtracting $6x$, there remains $xx = 91 - 6x$, and the rule gives $x = -3 \pm \sqrt{9 + 91} = -3 \pm 10$; so that $x = 7$, and $x = -13$.

Answer. The question admits of two solutions;

By one, the less number x is $= 7$, and the greater $x + 6 = 13$.

By the other, the less number $x = -13$, and the greater $x + 6 = -7$.

564. *Question* II. To find a number such, that if 9 be taken from its square, the remainder may be a number, as many units greater than 100, as the number sought is less than 23.

Let the number sought $= x$; we know, that $xx - 9$ exceeds 100 by $xx - 109$. And since x is less than 23 by $23 - x$, we have this equation; $xx - 109 = 23 - x$.

Wherefore $xx = -x + 132$, and, by the rule,

$$x = -\tfrac{1}{2} \pm \sqrt{\frac{1}{4} + 132} = -\frac{1}{2} \pm \sqrt{\frac{529}{4}} = -\frac{1}{2} \pm \frac{23}{2}.$$

So that $x = 11$, and $x = -12$.

Answer. When only a positive number is required, that number will be 11, the square of which *minus* 9 is 112, and consequently greater than 100 by 12, in the same manner as 11 is less than 23 by 12.

565. *Question* III. To find a number such, that if we multiply its half by its third, and to the product add half the number required, the result will be 30.

Suppose that number $= x$, its half, multiplied by its third, will make $\frac{1}{6}xx$; so that $\frac{1}{6}xx + \frac{1}{2}x = 30$. Multiplying by 6, we have $xx + 3x = 180$, or $xx = -3x + 180$, which gives $x = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + 180} = -\frac{3}{2} \pm \frac{27}{2}$.

Consequently x is either $= 12$, or -15.

566. *Question* IV. To find two numbers in a double ratio to each other, and such that if we add their sum to their product, we may obtain 90.

Let one of the numbers $= x$, then the other will be $= 2x$; their product also $= 2xx$, and if we add to this $3x$, or their sum, the new sum ought to make 90. So that $2xx + 3x = 90$; $2xx = 90 - 3x$; $xx = -\frac{3}{2}x + 45$, whence we obtain

$$x = -\tfrac{3}{4} \pm \sqrt{\frac{9}{16} + 45} = -\frac{3}{4} \pm \frac{27}{4}.$$

Consequently $x = 6$, or $-7\frac{1}{2}$.

567. *Question* V. A horse dealer, who bought a horse for a certain number of crowns, sells it again for 119 crowns, and his profit is as much per cent. as the horse cost him. Required, what he gave for it?

Suppose the horse cost x crowns; then as the horse dealer gains x per cent. we shall say, if 100 give the profit x; what does x give? *Answer*, $\frac{xx}{100}$. Since, therefore, he has gained $\frac{xx}{100}$, and the horse originally cost him x crowns, he must have sold it for $x + \frac{xx}{100}$; wherefore $x + \frac{xx}{100} = 119$. Subtracting x, we have $\frac{xx}{100} = -x + 119$; and multiplying by 100, we have $xx = -100x + 11900$. Applying the rule, we find $x = -50 \pm \sqrt{2500 + 11900} = -50 \pm \sqrt{14400} = -50 \pm 120$.

Answer. The horse cost 70 crowns, and since the horse dealer gained 70 per cent. when he sold it again, the profit must have been 49 crowns. The horse must have been, therefore, sold again for $70 + 49$, that is to say, for 119 crowns.

568. *Question* VI. A person buys a certain number of pieces of cloth; he pays, for the first, 2 crowns; for the second, 4 crowns; for the third, 6 crowns, and in the same manner always

2 crowns more for each following piece. Now, all the pieces together cost him 110. How many pieces had he?

Let the number sought $= x$. By the question the purchaser paid for the different pieces of cloth in the following manner;

for the $1, 2, 3, 4, 5 \ldots\ldots x$

he pays $2, 4, 6, 8, 10 \ldots\ldots 2x$ crowns.

It is therefore required to find the sum of the arithmetical progression $2 + 4 + 6 + 8 + 10 + \ldots\ldots 2x$, which consists of x terms, that we may deduce from it the price of all the pieces of cloth taken together. The rule which we have already given for this operation, requires us to add the last term and the first; the sum of which is $2x + 2$; if we multiply this sum by the number of terms x, the product will be $2xx + 2x$; if we lastly divide by the difference 2, the quotient will be $xx + x$, which is the sum of the progression; so that we have $xx + x = 110$; wherefore $xx = -x + 110$,

$$\text{and } x = -\frac{1}{2} + \sqrt{\frac{1}{4} + 110} = -\frac{1}{2} + \frac{21}{2} = 10.$$

Answer. The number of pieces of cloth is 10.

569. *Question* VII. A person bought several pieces of cloth, for 180 crowns. If he had received for the same sum 3 pieces more, he would have paid three crowns less for each piece; How many pieces did he buy?

Let us make the number sought $= x$; then each piece will have cost him $\frac{180}{x}$ crowns. Now, if the purchaser had had $x + 3$ pieces for 180 crowns, each piece would have cost $\frac{180}{x+3}$ crowns; and, since this price is less than the real price by three crowns, we have this equation,

$$\frac{180}{x+3} = \frac{180}{x} - 3.$$

Multiplying by x, we have $\frac{180x}{x+3} = 180 - 3x$; dividing by 3, we have $\frac{60x}{x+3} = 60 - x$; multiplying by $x + 3$ we have $60x = 180 + 57x - xx$; adding xx, we shall have $xx + 60x$

$= 180 + 57\,x$; subtracting $60\,x$, we shall have $x\,x = -3\,x + 180$.

The rule, consequently gives

$$x = -\frac{3}{2} + \sqrt{\frac{9}{4} + 180}, \text{ or } x = -\frac{3}{2} + \frac{27}{2} = 12.$$

Answer. He bought for 180 crowns 12 pieces of cloth at 15 crowns the piece, and if he had got 3 pieces more, namely 15 pieces for 180 crowns, each piece would have cost only 12 crowns, that is to say, 3 crowns less.

570. *Question* VIII. Two merchants enter into partnership with a stock of 100 crowns; one leaves his money in the partnership for three months, the other leaves his for two months, and each takes out 99 crowns of capital and profit. What proportion of the stock did each furnish?

Suppose the first partner contributed x crowns, the other will have contributed $100 - x$. Now, the former receiving 99 crowns, his profit is $99 - x$, which he has gained in three months with the principal x; and since the second receives also 99 crowns, his profit is $x - 1$, which he has gained in two months with the principal $100 - x$; it is evident also, that the profit of this second partner would have been $\frac{3\,x - 3}{2}$, if he had remained three months in the partnership. Now, as the profits gained in the same time are in proportion to the principals, we have the following proportion, $x : 99 - x = 100 - x : \frac{3\,x - 3}{2}$.

The equality of the product of the extremes to that of the means, gives the equation,

$$\frac{3\,x\,x - 3\,x}{2} = 9900 - 199\,x + x\,x;$$

Multiplying by 2, we have $3\,x\,x - 3\,x = 19800 - 398\,x + 2\,x\,x$; subtracting $2\,x\,x$, we have $x\,x - 3\,x = 19800 - 398\,x$ adding $3\,x$, we have $x\,x = 19800 - 395\,x$.

Wherefore by the rule,

$$x = -\frac{395}{2} + \sqrt{\frac{156025}{4} + \frac{79200}{4}} = -\frac{395}{2} + \frac{485}{2} = \frac{90}{2} = 45.$$

Answer. The first partner contributed 45 crowns, and the other 55 crowns. The first, having gained 54 crowns in three months, would have gained in one month 18 crowns; and the second having gained 44 crowns in two months, would have gained 22 crowns in one month: now these profits agree; for, if with 45 crowns 18 crowns are gained in one month, 22 crowns will be gained in the same time with 55 crowns.

571. *Question* IX. Two girls carry 100 eggs to market; one had more than the other, and yet the sum which they both received for them was the same. The first says to the second, if I had had your eggs, I should have received 15 sous. The other answers, if I had had yours, I should have received $6\frac{2}{3}$ sous. How many eggs did each carry to market?

Suppose the first had x eggs; then the second must have had $100 - x$.

Since therefore the former would have sold $100 - x$ eggs for 15 sous, we have the following proportion;

$$100 - x : 15 = x \ldots \text{to } \frac{15x}{100 - x} \text{ sous.}$$

Also, since the second would have sold x eggs for $6\frac{2}{3}$ sous, we find how much she got for $100 - x$ eggs, by saying

$$x : \frac{20}{3} = 100 - x \ldots \text{to } \frac{2000 - 20x}{3x}.$$

Now both the girls received the same money; we have consequently the equation, $\frac{15x}{100 - x} = \frac{2000 - 20x}{3x}$, which becomes this,

$$25xx = 200000 - 4000x;$$

and lastly this,

$$xx = -160x + 8000;$$

whence we obtain

$$x = -80 + \sqrt{6400 + 8000} = -80 + 120 = 40.$$

Answer. The first girl had 40 eggs, the second had 60, and each received 10 sous.

572. *Qustion* X. Two merchants sell each a certain quantity of stuff; the second sells 3 ells more than the first, and they received together 35 crowns. The first says to the second, I should have got 24 crowns for your stuff; the other answers, and I should have got for yours 12 crowns and a half. How many ells had each?

Suppose the first had x ells; then the second must have had

$x+3$ ells. Now, since the first would have sold $x+3$ ells for 24 crowns, he must have received $\frac{24x}{x+3}$ crowns for his x ells. And with regard to the second, since he would have sold x ells for $12\frac{1}{2}$ crowns, he must have sold his $x+3$ ells for $\frac{25x+75}{2x}$; so that the whole sum they received was $\frac{24x}{x+3}+\frac{25x+75}{2x}=35$ crowns.

This equation becomes $xx=20x-75$, whence we have $x=10\pm\sqrt{100-75}=10\pm5$.

Answer. The question admits of two solutions; according to the first, the first merchant had 15 ells, and the second had 18; and since the former would have sold 18 ells for 24 crowns, he must have sold his 15 ells for 20 crowns; the second, who would have sold 15 ells for 12 crowns and a half, must have sold his 18 ells for 15 crowns; so that they actually received 35 crowns for their commodity.

According to the second solution, the first merchant had 5 ells, and the other 8 ells; so that, since the first would have sold 8 ells for 24 crowns, he must have received 15 crowns for his 5 ells; and since the second would have sold 5 ells for 12 crowns and a half, his 8 ells must have produced him 20 crowns. The sum is, as before, 35 crowns.

CHAPTER VII.

Of the Nature of Equations of the Second Degree.

573. WHAT we have already said sufficiently shows, that equations of the second degree admit of two solutions; and this property ought to be examined in every point of view, because the nature of equations of a higher degree will be very much illustrated by such an examination. We shall therefore retrace, with more attention, the reasons which render an equation of the second degree capable of a double solution; since they undoubtedly will exhibit an essential property of those equations.

574. We have already seen, it is true, that this double solution arises from the circumstance that the square root of any number may be taken either positively, or negatively; however, as this principle will not easily apply to equations of higher degrees, it may be proper to illustrate it by a distinct analysis. Taking, for an example, the quadratic equation, $xx = 12x - 35$, we shall give a new reason for this equation being resolvible in two ways, by admitting for x the values 5 and 7, both of which satisfy the terms of the equation.

575. For this purpose it is most convenient to begin with transposing the terms of the equation, so that one of the sides may become 0; this equation consequently takes the form $xx - 12x + 35 = 0$; and it is now required to find a number such, that, if we substitute it for x, the quantity $xx - 12x + 35$ may be really equal to nothing; after this, we shall have to show how this may be done in two ways.

576. Now, the whole of this consists in showing clearly, that *a quantity of the form* $xx - 12x + 35$ *may be considered as the product of two factors*; thus, in fact, the quantity of which we speak is composed of the two factors $(x - 5) \times (x - 7)$. For, since this quantity must become 0, we must also have the product $(x - 5) \times (x - 7) = 0$; but *a product, of whatever number of factors it is composed, becomes* $= 0$, *only when one of those factors is reduced to* 0; this is a fundamental principle to which we must pay particular attention, especially when equations of several degrees are treated of.

577. It is therefore easily understood, that *the product* $(x - 5) \times (x - 7)$ *may become* 0 *in two ways: one, when the first factor* $x - 5 = 0$; *the other, when the second factor* $x - 7 = 0$. *In the first case* $x = 5$, *in the other*, $x = 7$. The reason is, therefore, very evident, why such an equation $xx - 12x + 35 = 0$, admits of two solutions, that is to say, why we can assign two values of x, both of which equally satisfy the terms of the equation. This fundamental principle consists in this, that the quantity $xx - 12x + 35$ may be represented by the product of two factors.

578. The same circumstances are found in all equations of the second degree. For, after having brought all the terms to

one side, we always find an equation of the following form $xx - ax + b = 0$, and this formula may be always considered as the product of two factors, which we shall represent by $(x - p) \times (x - q)$, without concerning ourselves what numbers the letters p and q represent. Now, as this product must be $= 0$, from the nature of our equation it is evident that this may happen in two ways; in the first place, when $x = p$; and in the second place, when $x = q$; and these are the two values of x which satisfy the terms of the equation.

579. Let us now consider the nature of these two factors, in order that the multiplication of the one by the other may exactly produce $xx - ax + b$. By actually multiplying them, we get $xx - (p + q)x + pq$; now this quantity must be the same as $xx - ax + b$, wherefore we have evidently $p + q = a$, and $pq = b$. So that we have deduced this very remarkable property, that *in every equation of the form* $xx - ax + b = 0$, *the two values of* x *are such, that their sum is equal to* a, *and their product equal to* b; *whence it follows that, if we know one of the values, the other also is easily found.*

580. We have considered the case in which the two values of x are positive, and which requires the second term of the equation to have the sign $-$, and the third term to have the sign $+$. Let us also consider the cases in which either one or both values of x become negative. The first takes place when the two factors of the equation give a product of this form $(x - p) \times (x + q)$; for then the two values of x are $x = p$, and $x = -q$; the equation itself becomes $xx + (q - p)x - pq = 0$; the second term has the sign $+$, when q is greater than p, and the sign $-$, when q is less than p; lastly, the third term is always negative.

The second case, in which both values of x are negative, occurs, when the two factors are $(x + p) \times (x + q)$; for we shall then have $x = -p$ and $x = -q$; the equation itself becomes $xx + (p + q)x + pq = 0$, in which both the second and third terms are affected by the sign $+$.

581. The signs of the second and the third term consequently show us the nature of the roots of any equation of the second degree. Let the equation be $xx \ldots . ax \ldots . b = 0$, if the

second and third terms have the sign $+$, the two values of x are both negative; if the second term has the sign $-$, and the third term has $+$, both values are positive; lastly, if the third term also has the sign $-$, one of the values in question is positive. But in all cases, whatever, the second term contains the sum of the two values, and the third term contains their product.

582. After what has been said, it will be very easy to form equations of the second degree containing any two given values Let there be required, for example, an equation such, that one of the values of x may be 7, and the other -3. We first form the simple equations $x=7$ and $x=-3$; thence these, $x-7=0$ and $x+3=0$, which gives us, in this manner, the factors of the equation required, which consequently becomes $xx-4x-21=0$. Applying here, also, the above rule, we find the two given values of x; for if $xx=4x+21$, we have $x=2\pm\sqrt{25}=2\pm 5$, that is to say, $x=7$, or $x=-3$.

583. The values of x may also happen to be equal. Let there be sought, for example, an equation, in which both values may be $=5$. The two factors will be $(x-5)\times(x-5)$, and the equation sought will be $xx-10x+25=0$. In this equation, x appears to have only one value; but it is because x is twice found $=5$, as the common method of resolution shows; for we have $xx=10x-25$; wherefore $x=5\pm\sqrt{0}=5\pm 0$, that is to say, x is in two ways $=5$.

584. A very remarkable case, in which both values of x become imaginary, or impossible, sometimes occurs; and it is then wholly impossible to assign any value for x, that would satisfy the terms of the equation. Let it be proposed, for example, to divide the number 10 into two parts, such, that their product may be 30. If we call one of those parts x, the other will be $=10-x$, and their product will be $10x-xx=30$; wherefore $xx=10x-30$, and $x=5\pm\sqrt{-5}$, which being *an imaginary number, shews that the question is impossible.*

585. It is very important, therefore, to discover some sign, by means of which we may immediately know, whether an equation of the second degree is possible or not.

Let us resume the general equation $ax-xx+b=0$.

We shall have $xx = ax - b$, and $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - b}$.

This shows, that if b is greater than $\frac{1}{4}aa$, or $4b$ greater than aa, the two values of x are always imaginary, since it would be required to extract the square root of a negative quantity; on the contrary, if b is less than $\frac{1}{4}aa$, or even less than 0, that is to say, is a negative number, both values will be possible or real. But whether they be real or imaginary, it is no less true, that they are still expressible, and always have this property, that their sum is $= a$, and their product $= b$. In the equation $xx - 6x + 10 = 0$, for example, the sum of the two values of x must be $= 6$, and the product of these two values must be $= 10$; now we find, I. $x = 3 + \sqrt{-1}$, and II. $x = 3 - \sqrt{-1}$, quantities whose sum $= 6$, and the product $= 10$.

586. The expression, which we have just found, may be represented in a manner more general, and so as to be applied to equations of this form, $fxx \pm gx + h = 0$; for this equation gives $xx = \pm \frac{gx}{f} - \frac{h}{f}$, and $x = \pm \frac{g}{2f} \pm \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$, or $x = \frac{\pm g \pm \sqrt{gg - 4fh}}{2f}$; whence we conclude that the two values are imaginary, and consequently the equation impossible, when $4fh$ is greater than gg; that is to say, when, in the equation $fxx - gx + h = 0$, four times the product of the first and the last term exceeds the square of the second term: for the product of the first and the last term, taken four times, is $4fhxx$, and the square of the middle term is $ggxx$; now, if $4fhxx$ is greater than $ggxx$, $4fh$ is also greater than gg, and in that case, the equation is evidently impossible. In all other cases the equation is possible, and two real values of x may be assigned. It is true they are often irrational; but we have already seen, that, in such cases, we may always find them by approximation; whereas no approximations can take place with regard to imaginary expressions, such as $\sqrt{-5}$; for 100 is as far from being the value of that root, as 1, or any other number.

587. We have further to observe, that *any quantity of the second degree,* x x $\pm$ a x $\pm$ b, *must always be resolvible into two factors,* such as $(x \pm p) \times (x \pm q)$. For, if we took three

factors, such as these, we should come to a quantity of the third degree, and taking only one such factor, we should not exceed the first degree.

It is therefore certain that *every equation of the second degree necessarily contains two values of x, and that it can neither have more nor less.*

588. We have already seen, that when the two factors are found, the two values of x are also known, since each factor gives one of those values, when it is supposed to be $=0$. The converse also is true, *viz.* that when we have found one value of x, we know also one of the factors of the equation; for if $x=p$ represents one of the values of x, in any equation of the second degree, $x-p$ is one of the factors of that equation; that is to say, all the terms having been brought to one side, the equation is divisible by $x-p$; and further, the quotient expresses the other factor.

589. In order to illustrate what we have now said, let there be given the equation $xx+4x-21=0$, in which we know that $x=3$ is one of the values of x, because $\overline{3\times 3}) + \overline{4\times 3}) -21=0$; this shows, that $x-3$ is one of the factors of the equation, or that $xx+4x-21$ is divisible by $x-3$, which the actual division proves.

$$
\begin{array}{r}
x-3)\ xx+4x-21\ (x+7 \\
xx-3x \qquad\quad \\
\hline
7x-21 \\
7x-21 \\
\hline
0.
\end{array}
$$

So that the other factor is $x+7$, and our equation is represented by the product $(x-3)\times(x+7)=0$; whence the two values of x immediately follow, the first factor giving $x=3$, and the other $x=-7$.

QUESTIONS FOR PRACTICE.

Fractions.

SECTION I. CHAPTER 9.

1. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to a common denominator.

Ans. $\frac{2cx}{ac}$ and $\frac{ab}{ac}$.

2. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to a common denominator.

Ans. $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

3. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d to fractions having a common denominator.

Ans. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$ and $\frac{6acd}{6ac}$.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{2x}{a}$ to a common denominator.

Ans. $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2+24x}{12a}$,

5. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$, and $\frac{x^2+a^2}{x+a}$ to a common denominator.

Ans. $\frac{3x+3a}{6x+6a}$, $\frac{2a^2x+2a^3}{6x+6a}$, $\frac{6x^2+6a^2}{6x+6a}$,

6. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$, and $\frac{d}{a}$, to a common denominator.

Ans. $\frac{2a^2b}{4a^4}$, $\frac{2a^3c}{4a^4}$, and $\frac{4a^3d}{4a^4}$.

SECTION I. CHAPTER 10.

7. Required the product of $\frac{x}{6}$ and $\frac{2x}{9}$. *Ans.* $\frac{x^2}{27}$.

8. Required the product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$. *Ans.* $\frac{4x^3}{21}$.

9. Required the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$. *Ans.* $\frac{x^2+ax}{a^2+ac}$.

10. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. *Ans.* $\frac{9ax}{2b}$.

11. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. *Ans.* $\frac{3x^3}{5a}$.

12. Required the product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$. *Ans.* $9ax$.

13. Required the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$. *Ans.* $\frac{ab+bx}{x}$.

14. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

Ans. $\frac{x^4-b^4}{b^2c+bc^2}$.

15. Required the product of x, $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

Ans. $\frac{x^3-x}{a^2+ab}$.

16. Required the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$. *Ans.* $1\frac{1}{2}$.

17. Required the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$. *Ans.* $\frac{ad}{2bc}$.

18. Required the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

Ans. $\frac{5x^2+6ax+a^2}{2x^2-2b^2}$.

19. Required the quotient of $\frac{2x^2}{a^3+x^3}$ divided by $\frac{x}{x+a}$.

Ans. $\frac{2x}{x^2-ax+a^2}$.

20. Required the quotient of $\frac{7x}{5}$ divided by $\frac{12}{13}$. *Ans.* $\frac{91x}{60}$.

21. Required the quotient of $\frac{4x^2}{7}$ divided by $5x$. *Ans.* $\frac{4x}{35}$.

22. Required the quotient of $\frac{x+1}{6}$ divided by $\frac{2x}{3}$. *Ans.* $\frac{x+1}{4x}$.

23. Required the quotient of $\frac{x-b}{8cd}$ divided by $\frac{3cx}{4d}$. *Ans.* $\frac{x-b}{6c^2x}$.

24. Required the quotient of $\frac{x^4-b^4}{x^2-2bx+b^2}$ divided by $\frac{x^2+bx}{x-b}$.

Ans. $x+\frac{b^2}{x}$.

Infinite Series.

SECTION II. CHAPTER 5.

25. Resolve $\frac{ax}{a-x}$ into an infinite series.

Ans. $x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}$, &c.

26. Resolve $\frac{b}{a+x}$ into an infinite series.

Ans. $\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+$ &c.

or resolved into factors,

$\frac{b}{a}\times(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$ &c.)

27. Resolve $\frac{a^2}{x+b}$ into an infinite series.

Ans. $\frac{a^2}{x}\times(1-\frac{b}{x}+\frac{b^2}{x^2}-\frac{b^3}{x^3}+$ &c.)

28. Resolve $\frac{1+x}{1-x}$ into an infinite series.

Ans. $1+2x+2x^2+2x^3+2x^4$, &c.

29. Resolve $\frac{a^2}{(a+x)^2}$ into an infinite series.

Ans. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}$, &c.

Surds or Irrational Numbers.

SECTION I. CHAPTERS 12, 19; and SECTION II. CHAPTER 8, &c.

30. Reduce 6 to the form of $\sqrt{5}$. *Ans.* $\sqrt{36}$.

31. Reduce $a+b$ to the form of $\sqrt{bc}$. *Ans.* $\sqrt{aa+2ab+bb}$.

32. Reduce $\frac{a}{b\sqrt{c}}$ to the form of $\sqrt{d}$. *Ans.* $\sqrt{\frac{aa}{bbc}}$

33. Reduce a^2 and $b^{\frac{3}{2}}$ to the common exponent $\frac{1}{3}$.

Ans. $\overline{a^{6}}|^{\frac{1}{3}}$, and $\overline{b^{\frac{9}{2}}}|^{\frac{1}{3}}$.

34. Reduce $\sqrt{48}$ to its simplest form. *Ans.* $4\sqrt{3}$.

35 Reduce $\sqrt{a^3x-a^2x^2}$ to its simplest form.

Ans. $a\sqrt{ax-xx}$.

36. Reduce $\sqrt[3]{\frac{27a^4b^3}{8b-8a}}$ to its simplest form.

Ans. $\frac{3ab}{2}\sqrt[3]{\frac{a}{b-a}}$.

37. Add $\sqrt{6}$ to $2\sqrt{6}$; and $\sqrt{8}$ to $\sqrt{50}$. *Ans.* $3\sqrt{6}$; and $\sqrt{98}$.

38. Add $\sqrt{4a}$ and $\sqrt[4]{a^6}$ together. *Ans.* $(a+2)\sqrt{a}$.

39. Add $\overline{\frac{b}{c}}|^{\frac{1}{2}}$ and $\overline{\frac{c}{b}}|^{\frac{3}{2}}$ together. *An.* $\frac{bb+cc}{b\sqrt{bc}}$.

40. Subtract $\sqrt{4a}$ from $\sqrt[4]{a^6}$. *Ans.* $(a-2)\sqrt{a}$.

41. Subtract $\overline{\frac{c}{b}}|^{\frac{3}{2}}$ from $\overline{\frac{b}{c}}|^{\frac{1}{2}}$. *Ans.* $\frac{bb-cc}{b}\sqrt{\frac{1}{bc}}$.

42. Multiply $\sqrt{\frac{2ab}{3c}}$ by $\sqrt{\frac{9ad}{2b}}$. *Ans.* $\sqrt{\frac{3a^2d}{c}}$.

43. Multiply $\sqrt{d}$ by $\sqrt[3]{ab}$. *Ans.* $\sqrt[6]{a^2b^2d^3}$.

44. Multiply $\sqrt{4a-3x}$ by $2a$. *Ans.* $\sqrt{16a^3-12a^2x}$.

45. Multiply $\frac{a}{2b}\sqrt{a-x}$ by $(c-d)\sqrt{ax}$.

Ans. $\frac{ac-ad}{2b}\sqrt{a^2x-ax^2}$.

46 Multiply $\sqrt{a}-\sqrt{b}-\sqrt{3}$ by $\sqrt{a}+\sqrt{b}-\sqrt{3}$.

Ans. $\sqrt{a^2-b}+\sqrt{3}$.

47. Divide $a^{\frac{2}{3}}$ by $a^{\frac{1}{4}}$; and $a^{\frac{1}{n}}$ by $a^{\frac{1}{m}}$. *Ans.* $a^{\frac{5}{12}}$ and $a^{\frac{m-n}{mn}}$.

48. Divide $\frac{ac-ad}{2b}\sqrt{a^2x-ax^2}$ by $\frac{a}{2b}\sqrt{a-x}$.

Ans. $(c-d)\sqrt{ax}$.

49. Divide $a^2-ad-b+d\sqrt{b}$ by $a-\sqrt{b}$.

Ans. $a+\sqrt{b}-d$.

50. What is the cube of $\sqrt{2}$? *Ans.* $\sqrt{8}$.

51. What is the square of $3\sqrt[3]{bc^2}$? *Ans.* $9c\sqrt[3]{b^2c}$.

52. What is the fourth power of $\frac{a}{2b}\sqrt{\frac{2a}{c-b}}$?

Ans. $\frac{a^6}{4b^4(c^2-2bc+b^2)}$.

53. What is the square of $3+\sqrt{5}$? *Ans.* $14+6\sqrt{5}$.

54. What is the square root of a^3? *Ans.* $a^{\frac{3}{2}}$; or $\sqrt{a^3}$.

55. What is the cube root of ab^2? *Ans.* $a\,b\,b^{\frac{1}{3}}$; or $\sqrt[3]{ab\,b}$.

56. What is the cube root of $\sqrt{a^2-x^2}$? *Ans.* $\sqrt[6]{a^2-x^2}$.

57. What is the cube root of $a^2-\sqrt{ax-x^2}$?

Ans. $\sqrt[3]{a^2-\sqrt{ax-x^2}}$.

58. What multiplier will render $a+\sqrt{3}$ rational?

Ans. $a-\sqrt{3}$.

59. What multiplier will render $\sqrt{a}-\sqrt{b}$ rational?

Ans. $\sqrt{a}+\sqrt{b}$.

60. What multiplier will render the denominator of the fraction $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$ rational? *Ans.* $\sqrt{7}-\sqrt{3}$.

SECTION II. CHAPTER 12.

61. Resolve $\sqrt{a^2+x^2}$ into an infinite series.

Ans. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{828a^7}$, &c.

62. Resolve $\sqrt{1+1}$ into an infinite series.

$$\textit{Ans. } 1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}, \text{ \&c.}$$

63. Resolve $\sqrt{a^2-x^2}$ into an infinite series.

$$\textit{Ans. } a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}, \text{ \&c.}$$

64. Resolve $\sqrt[3]{1-x^3}$ into an infinite series.

$$\textit{Ans. } 1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}, \text{ \&c.}$$

65. Resolve $\sqrt{r^2-x^2}$ into an infinite series.

$$\textit{Ans. } r-\frac{x^2}{2r}-\frac{x^4}{8r^3}-\frac{x^6}{16r^5}-\frac{5x^8}{128r^7}, \text{ \&c.}$$

66. Resolve $\frac{1}{\sqrt{a^2-x^2}}$ into an infinite series.

$$\textit{Ans. } \frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}+\frac{15x^6}{48a^7}, \text{ \&c.}$$

67. Resolve $(a^2-x^2)^{\frac{1}{5}}$ into an infinite series.

$$\textit{Ans. } a^{\frac{2}{5}}\times\left(1-\frac{x^2}{5a^2}-\frac{2x^4}{25a^4}-\frac{6x^6}{125a^6}-\text{ \&c.}\right.$$

68. Resolve $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$ into an infinite series.

$$\textit{Ans. } 1+\frac{x^2}{a^2}+\frac{x^4}{2a^4}+\frac{x^6}{2a^6}, \text{ \&c.}$$

69. Resolve $\sqrt[3]{\frac{a^2+x^2}{(a^2+x^2)^3}}$ into an infinite series.

$$\textit{Ans. } \frac{1}{a\sqrt[3]{a}}\times\left(1-\frac{2x^2}{3a^2}+\frac{5x^4}{9a^4}-\frac{40x^6}{81a^6}+, \text{ \&c.}\right.$$

Summation of Arithmetical Progressions.

SECTION III. CHAPTER 4.

70. Required the sum of an increasing arithmetical progression, having 3 for its first term, 2 for the common difference, and the number of terms 20. *Ans.* 440.

71. Required the sum of a decreasing arithmetical progres-

sion, having 10 for its first term, $\frac{1}{3}$ for the common difference, and the number of terms 21. *Ans.* 140.

72. Required the number of all the strokes of a clock in twelve hours, that is, a complete revolution of the index.

Ans. 78.

73. The clocks of Italy go on to 24 hours; how many strokes do they strike in a complete revolution of the index? *Ans.* 300.

74. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other, how far will a person travel who shall bring them one by one to a basket, which is placed one yard from the first stone.

Ans. 5 *miles* and 1300 *yards*.

The greatest Common Divisor.

SECTION III. CHAPTER 6.—SECTION I. CHAPTER 8.

75. Reduce $\frac{cx + x^2}{ca^2 + a^2 x}$ to its lowest terms. *Ans.* $\frac{x}{a^2}$.

76. Reduce $\frac{x^3 - b^2 x}{x^2 + 2bx + b^2}$ to its lowest terms. *Ans.* $\frac{x^2 - bx}{x + b}$.

77. Reduce $\frac{x^4 - b^4}{x^5 - b^2 x^3}$ to its lowest terms. *Ans.* $\frac{x^2 + b^2}{x^3}$.

78. Reduce $\frac{x^2 - y^2}{x^4 - y^4}$ to its lowest terms. *Ans.* $\frac{1}{x^2 + y^2}$.

79. Reduce $\frac{a^4 - x^4}{a^3 - a^2 x + a x^2 - x^3}$ to its lowest terms. *Ans.* $\frac{a + x}{1}$.

80. Reduce $\frac{5a^5 + 10a^4 x + 5a^3 x^2}{a^3 x + 2a^2 x^2 + 2a x^3 + x^4}$ to its lowest terms.

Ans. $\frac{5a^4 + 5a^3 x}{a^2 x + a x^2 + x^3}$.

Summation of Geometrical Progressions.

SECTION III. CHAPTER 10.

81. A SERVANT agreed with a master to serve him eleven years without any other reward for his service than the produce of one wheat corn for the first year; and that product to be sown the second year, and so on from year to year till the

end of the time, allowing the increase to be only in a tenfold proportion. What was the sum of the whole produce?

Ans. 11111111110 wheat corns.

N. B. It is further required, to reduce this number of corns to the proper measures of capacity, and then by supposing an average price of wheat to compute the value of the corns in money.

82. A servant agreed with a gentleman to serve him twelve months, provided he would give him a farthing for his first month's service, a penny for the second, and 4*d.* for the third, &c. What did his wages amount to? *Ans.* 5825*l.* 8*s.* $5\frac{1}{4}$*d.*

83. *Sessa*, an *Indian*, having invented the game of chess, showed it to his prince, who was so delighted with it, that he promised him any reward he should ask; upon which Sessa requested that he might be allowed one grain of wheat for the first square on the chess board, two for the second, and so on, doubling continually, to 64, the whole number of squares; now supposing a pint to contain 7680 of those grains, and one quarter to be worth 1*l.* 7*s.* 6*d.* it is required to compute the value of the whole sum of grains. *Ans.* £64481488296.

Simple Equations.

SECTION IV. CHAPTER 2.

84. If $x - 4 + 6 = 8$, then will $x = 6$.

85. If $4x - 8 = 3x + 20$, then will $x = 28$.

86. If $ax = ab - a$, then will $x = b - 1$.

87. If $2x + 4 = 16$, then will $x = 6$.

88. If $ax + 2ba = 3c^2$, then will $x = \frac{3c^2}{a} - 2b$.

89. If $\frac{x}{2} = 5 + 3$, then will $x = 16$.

90. If $\frac{2x}{3} - 2 = 6 + 4$, then will $x = 18$.

91. If $a - \frac{b}{x} = c$, then will $x = \frac{b}{a - c}$.

92. If $5x - 15 = 2x + 6$, then will $x = 7$.

93. If $40 - 6x - 16 = 120 - 14x$, then will $x = 12$.

94. If $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10$, then will $x = 24$.

95. If $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x-19}{2}$, then will $x = 23\frac{1}{4}$.

96. If $\sqrt{\frac{2}{3}x + 5} = 7$, then will $x = 6$.

97. If $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$, then will $x = a\sqrt{\frac{1}{3}}$.

98. If $3ax + \frac{a}{2} - 3 = bx - a$, then will $x = \frac{6-3a}{6a-2b}$.

99. If $\sqrt{12+x} = 2 + \sqrt{x}$, then will $x = 4$.

100. If $y + \sqrt{a^2 + y^2} = \frac{2a^2}{(a^2+y^2)^{\frac{1}{2}}}$, then will $y = \frac{1}{3}a\sqrt{3}$.

101. If $\frac{y+1}{2} + \frac{y+2}{3} = 16 - \frac{y+3}{4}$, then will $y = 13$.

102. If $\sqrt{x} + \sqrt{a+x} = \frac{2a}{\sqrt{a+x}}$, then will $x = \frac{a}{3}$.

103. If $\sqrt{a^2 + x^2} = \sqrt[4]{b^4 + x^4}$, then will $x = \sqrt{\frac{b^4 - a^4}{2a^2}}$.

104. If $x = \sqrt{a^2 + x\sqrt{b^2 + x^2}} - a$, then will $x = \frac{b^2}{4a} - a$.

105. If $\frac{128}{3x-4} = \frac{216}{5x-6}$, then will $x = 12$.

106. If $\frac{42x}{x-2} = \frac{35x}{x-3}$, then will $x - 8$.

107. If $\frac{45}{2x+3} = \frac{57}{4x-5}$, then will $x = 6$.

108. If $\frac{x^2-12}{3} = \frac{x^2-4}{4}$, then will $x = 6$.

109. If $615x - 7x^3 = 48x$, then will $x = 9$.

SECTION IV. CHAPTER 3.

110. To find a number, to which, if there be added a half, a third, and a fourth of itself, the sum will be 50. *Ans.* 24.

111. A person being asked what his age was, replied that $\frac{3}{4}$

of his age multiplied by $\frac{1}{12}$ of his age gives a product equal to his age. What was his age? *Ans.* 16.

112. The sum of 660*l.* was raised for a particular purpose by four persons, A, B, C, and D; B advanced twice as much A; C as much as A and B together; and D as much as B and C. What did each contribute? *Ans.* 60*l*, 120*l*, 180*l*, and 300*l*.

113. To find that number whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 12. *Ans.* 144.

114. What sum of money is that, whose $\frac{1}{3}$ part, $\frac{1}{4}$ part, and $\frac{1}{5}$ part added together, amount to 94 pounds? *Ans.* 120*l.*

115. In a mixture of copper, tin, and lead, one half of the whole — 16*lb.* was copper; $\frac{1}{3}$ of the whole — 12*lb.* tin; and $\frac{1}{4}$ of the whole + 4*lb.* lead: what quantity of each was there in the composition?

Ans. 128*lb.* of copper, 84*lb.* of tin, and 76*lb.* of lead.

116. What number is that, whose $\frac{1}{3}$ part exceeds its $\frac{1}{5}$ by 72? *Ans.* 540.

117. To find two numbers in the proportion of 2 to 1, so that if 4 be added to each, the two sums shall be in the proportion of 3 to 2. *Ans.* 8 and 4.

118. There are two numbers such that $\frac{1}{2}$ of the greater added to $\frac{1}{3}$ of the less is 13, and if $\frac{1}{2}$ of the less be taken from $\frac{1}{3}$ of the greater, the remainder is nothing; what are the numbers?

Ans. 18 and 12.

119. In the composition of a certain quantity of gunpowder $\frac{2}{3}$ of the whole plus 10 was *nitre*; $\frac{1}{6}$ of the whole minus $4\frac{1}{4}$ was *sulphur*, and the *charcoal* was $\frac{1}{7}$ of the nitre — 2. How many pounds of gunpowder were there? *Ans.* 69.

120. A person has a lease for 99 years; and being asked how much of it was already expired, answered, that two thirds of the time past was equal to four fifths of the time to come: required the time past. *Ans.* 54 years.

121. It is required to divide the number 48 into two such parts, that the one part may be three times as much above 20 as the other wants of 20. *Ans.* 32 and 16.

122. A person rents 25 acres of land at 7 pounds 12 shillings per annum; this land consisting of two sorts, he rents the better

sort at 8 shillings per acre, and the worse at 5: required the number of acres of the better sort. *Ans.* 9.

123. A certain cistern, which would be filled in 12 minutes by two pipes running into it, would be filled in 20 minutes by one alone. Required, in what time it would be filled by the other alone. *Ans.* 30 minutes.

124. Required two numbers, whose sum may be s, and their proportion as a to b. *Ans.* $\frac{as}{a+b}$ and $\frac{bs}{a+b}$.

125. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off making way at the rate of 8 miles an hour; it is demanded how many miles the ship can run before she will be overtaken? *Ans.* 72.

126. A gentleman distributing money among some poor people, found he wanted 10*s.* to be able to give 5*s.* to each; therefore he gives 4*s.* only, and finds that he has 5*s.* left: required the number of shillings and of poor people.

Ans. 15 poor people, and 65 shillings.

127. There are two numbers whose sum is the 6th part of their product, and the greater is to the less as 3 to 2. Required those numbers. *Ans.* 15 and 10.

N. B. This question may be solved likewise by means of one unknown letter.

128. To find three numbers, such that the first, with half the other two, the second with one third of the other two, and the third with one fourth of the other two, may be equal to 34.

Ans. 26, 22, and 10.

129. To find a number consisting of three places, whose digits are in arithmetical progression; if this number be divided by the sum of its digits, the quotient will be 48; and if from the number be subtracted 198, the digits will be inverted.

Ans. 432.

130. To find three numbers such, that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, equal to 38.

Ans. 24, 60, 120.

131. To find three numbers such that the first with $\frac{1}{2}$ of the

sum of the second and third shall be 120, the second with $\frac{1}{5}$ of the difference of the third and first shall be 70, and $\frac{1}{2}$ of the sum of the three numbers shall be 95. *Ans.* 50, 63, 75.

132. What is that fraction which will become equal to $\frac{1}{3}$, if an unit be added to the numerator; but on the contrary, if an unit be added to the denominator, it will be equal to $\frac{1}{4}$?

Ans. $\frac{4}{15}$.

133. The dimensions of a certain rectangular floor are such, that if it had been 2 feet broader, and 3 feet longer, it would have been 64 square feet larger; but if it had been 3 feet broader and 2 feet longer, it would then have been 68 square feet larger: required the length and breadth of the floor.

Ans. Length 14 feet, and breadth 10 feet.

134. A person found that upon beginning the study of his profession $\frac{1}{7}$ of his life hitherto had passed before he commenced his education, $\frac{1}{3}$ under a private teacher, and the same time at a public school, and four years at the university. What was his age? *Ans.* 21 years.

135. To find a number such that whether it be divided into two or three equal parts the continued product of the parts shall be equal to the same quantity. *Ans.* $6\frac{3}{4}$.

136. There is a certain number, consisting of two digits. The sum of these digits is 5, and if 9 be added to the number itself the digits will be inverted. What is the number?

Ans. 23.

137. What number is that, to which if I add 20 and from $\frac{2}{3}$ of this sum I subtract 12, the remainder shall be 10? *Ans.* 13.

Quadratic Equations.

SECTION IV. CHAPTER 5.

138. To find that number to which 20 being added, and from which 10 being subtracted, the square of the sum, added to twice the square of the remainder, shall be 17475. *Ans.* 75.

139. What two numbers are those, which are to one another in the ratio of 3 to 5, and whose squares, added together, make 1666? *Ans.* 21 and 35.

140. The sum $2a$, and the sum of the squares $2b$, of two numbers being given; to find the numbers.

Ans. $a - \sqrt{b - a^2}$ and $a + \sqrt{b - a^2}$.

141. To divide the number 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

142. To find three such numbers, that the sum of the first and second multiplied into the third, may be equal to 63; and the sum of the second and third, multiplied into the first equal to 28; also, that the sum of the first and third, multiplied into the second, may be equal to 55. *Ans.* 2, 5, 9.

143. What two numbers are those, whose sum is to the greater as 11 to 7; the difference of their squares being 132?

Ans. 14 and 8.